ASYMPTOTIC THEOREMS FOR ERROR STRUCTURES :

a) CENTRAL LIMIT THEOREM IN HILBERT SPACES

b) APPROXIMATIONS OF THE BROWNIAN MOTION (B.M) IN THE WIENER SPACE

Christophe Chorro, OSAKA, 2004/11/18

OPENING REMARKS :

As the normal law appears naturally in asymptotic probability theorems we will see in this talk that a family of error structures plays exactly the same role in the framework of Dirichlet forms.

Definition : Let B be a separable Banach space. We set

$\mathcal{A} = \{f(\lambda_1, .., \lambda_n); n \in \mathbb{N}, f \in C^1(\mathbb{R}^n) \cap Lip, (\lambda_1, .., \lambda_n) \in (B')^n\}$.

We will say that an error structure on B

$$S = (B, \mathcal{B}(B), m, \mathbb{D}, \Gamma)$$

is of the Ornstein-Uhlenbeck type if m is a gaussian measure on B, $\mathcal{A} \subset \mathbb{D}$ and for $F = f(\lambda_1, .., \lambda_n) \in \mathcal{A}$,

$$\Gamma[F] = \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} (\lambda_1, \dots, \lambda_n) \; a_{i,j} \qquad (1)$$

where the coefficients $a_{i,j}$ only depend on (λ_i, λ_j).

Examples :

- The Ornstein-Uhlenbeck structure on $\mathbb{R}$

$$(\mathbb{R}, \mathcal{B}(\mathbb{R}), m = \mathcal{N}(0,1), H^1(m), (u')^2).$$

- The Ornstein-Uhlenbeck structure of parameter c on the Wiener space $\mathcal{C} = (C_0([0,1], \mathbb{R}), \|\|_\infty)$ equipped with the Wiener measure ν :

$$S_{OU}^c = (\mathcal{C}, \mathcal{B}(\mathcal{C}), \nu, \mathbb{D}_{OU}, \Gamma_{OU}^c)$$

where Γ_{OU}^c is entirely determined by its action on the first Wiener chaos, $\forall f \in L^2([0,1], dx)$

$$\Gamma_{OU}^c \left[\int_0^1 h(s) dB_s \right] = c \int_0^1 h(s)^2 ds.$$

If $F = f(\lambda_1, .., \lambda_n)$ is a regular cylindrical function,

$$\Gamma_{OU}^c[F] = c \times \sum_{i,j=1}^n \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} (\lambda_1, \ldots, \lambda_n) < \lambda_i, \lambda_j >_{L^2([0,1],dx)} .$$

STUDY PLAN

I) Finite dimension

1) Image and Product of error structures
2) $\mathbb{D}$-independence
3) Convergence in $\mathbb{D}$-law
4) Central Limit Theorem in $\mathbb{R}^d$

II) Infinite dimension

1) Vectorial domain of a Dirichlet form in the sense of Feyel-La Pradelle.
2) Extension of the functional calculus and image of an error structure by an element of its vectorial domain.
3) Central limit theorem in Hilbert spaces.

III) Approximations of the Brownian motion in the Wiener space.

FINITE DIMENSION

1) Image and infinite products

• Image

Let $S = (W, \mathcal{W}, m, \mathbb{D}, \Gamma)$ be an error structure and $U \in \mathbb{D}^d$. We can see easily that the bilinear form

$$(C^1(\mathbb{R}^d) \cap Lip, F \mapsto \mathcal{E}[F(U)])$$

is closable and that its smallest closed extension $(\mathbb{D}_U, \mathcal{E}_U)$ is a local Dirichlet form with a squared field operator denoted by Γ_U. The error structure

$$U_* S = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), U_* m, \mathbb{D}_U, \Gamma_U)$$

is called the **image of S by U** or the **$\mathbb{D}$-law of U**.

Example : If F is the distribution function of a standard gaussian variable on $\mathbb{R}$,

$$F_*(\mathbb{R}, \mathcal{B}(\mathbb{R}), m = \mathcal{N}(0,1), H^1(m), (u')^2)$$

is called the **pseudo-gaussian** error structure on $[0, 1]$.

• Infinite products

Let $S_n = (W_n, \mathcal{W}_n, m_n, \mathbb{D}_n, \Gamma_n)$, $n \geq 0$, be a family of error structures. The product structure

$$(W, \mathcal{W}, m, \mathbb{D}, \Gamma) = \prod_{n=0}^{\infty} S_n$$

is defined by $(W, \mathcal{W}, m) = (\prod_{n=0}^{\infty} W_n, \prod_{n=0}^{\infty} \mathcal{W}_n, \otimes_{n=0}^{\infty} m_n)$ with an explicit domain $\mathbb{D}$ and $\forall F \in \mathbb{D}$,

$$\Gamma[F] = \sum_{n=0}^{\infty} \Gamma_n[F].$$

2) $\mathbb{D}$-independence

The random variables $U_1 \in \mathbb{D}^p$ and $U_2 \in \mathbb{D}^m$ are said to be $\mathbb{D}$-independent if

$$(U_1)_* S \otimes (U_2)_* S = (U_1, U_2)_* S.$$

3) Convergence in $\mathbb{D}$-law

We say that a sequence $(U_n)_{n\in\mathbb{N}} \in \mathbb{D}^p$ converges in $\mathbb{D}$-law if there exists an error structure $\tilde{S} = (\mathbb{R}^p, \mathcal{B}(\mathbb{R}^p), P, \tilde{\mathbb{D}}, \tilde{\Gamma})$ such that :
i) $(U_n)_* m \to P$ in distribution,
ii) $C^1(\mathbb{R}^p, \mathbb{R}) \cap Lip \subset \tilde{\mathbb{D}}$ and $\forall F \in C^1(\mathbb{R}^p, \mathbb{R}) \cap Lip$,

$$\mathcal{E}[F(U_n)] \underset{n\to\infty}{\longrightarrow} \tilde{\mathcal{E}}[F].$$

$$\Updownarrow$$

i),ii) and iii) the bilinear form $(C^1(\mathbb{R}^p, \mathbb{R}) \cap Lip, \tilde{\mathcal{E}})$ is closable and may be extended to a local Dirichlet form having a squared field operator.

For convenience, we will say that $(U_n)_{n\in\mathbb{N}} \in \mathbb{D}^p$ converges in $\mathbb{D}$-law toward $\tilde{S}$.

Remarks : • There are examples where i) and ii) are fulfilled but not iii).

• The limit is unique if one forces $\tilde{\mathbb{D}}$ to be minimal for inclusion.

• If (U_n) converges toward U in $\mathbb{D}$, then (U_n) converges in $\mathbb{D}$-law toward $U_* S$.

4) Central Limit Theorem in $\mathbb{R}^d$ (Bouleau-Hirsch)

Let (U_n) be a sequence in $\mathbb{D}^p$ of Dirichlet-independent random variables with the same Dirichlet-law. If we suppose that $U_1 = (U_1^1, ..., U_1^p)$ has zero mean and a covariance matrix denoted by Σ then, $V_n = \frac{U_1 + ... + U_n}{\sqrt{n}}$ converges in Dirichlet-law toward $\tilde{S} = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \nu, \tilde{\mathbb{D}}, \tilde{\Gamma})$ where

a) ν is a gaussian measure on $\mathbb{R}^d$ with zero mean and covariance matrix Σ,

b) $\forall F \in C^1(\mathbb{R}^p, \mathbb{R}) \cap Lip$,

$$\tilde{\Gamma}[F] = \sum_{i,j=1}^{p} \frac{\partial F}{\partial x_i} \frac{\partial F}{\partial x_j} a_{i,j}$$

where $a_{i,j} = \mathcal{E}[U_1^i, U_1^j]$.

Sketch of the proof :

i) Classical CLT in $\mathbb{R}^p$.

ii) By a direct computation.

iii) Using the classical integration by part formulae, $\forall F, G \in C^1(\mathbb{R}^p, \mathbb{R}) \cap Lip$

$$\tilde{\mathcal{E}}[F, G] = - < A[F], G >_{L^2(\nu)} . \square$$

INFINITE DIMENSION

In this section we suppose that $S = (W, \mathcal{W}, m, \mathbb{D}, \Gamma)$ is an error structure having a gradient :
There exists a separable Hilbert space $(\mathcal{H}, \|\|_{\mathcal{H}})$ and an operator ∇ from $\mathbb{D}$ into $L^2(m; \mathcal{H})$ such that

$$\forall X \in \mathbb{D} \qquad \| \nabla X \|_{\mathcal{H}}^2 = \Gamma[X].$$

Let B be a Banach space having a Schauder basis.

We would like to extend the notion of domain for $B-$valued random variables.

1) Vectorial domain of a Dirichlet form

Let us denote by J an isometry from $\mathcal{H}$ into $L^2((\tilde{W}, \tilde{\mathcal{W}}, \tilde{m}))$ such that $\forall h \in \mathcal{H}$, $\mathbb{E}_{\tilde{m}}[J(h)] = 0$. The sharp operator $\#$ is defined in the following way, $\forall U \in \mathbb{D}$,

$$U^{\#} = J(\nabla U) \in L^2(m \otimes \tilde{m}).$$

Definition (Feyel-La Pradelle) : Let us denote by $\mathbb{D}_B$ the vector space of random variables U in $L^2(m; B)$ such that there exists g in $L^2(m \otimes \tilde{m}; B)$ so that

$$\forall \lambda \in B', \; < \lambda, U > \in \mathbb{D} \;\; \textit{and} \;\; < \lambda, U >^{\#} = < \lambda, g > .$$

We then put $g = U^{\#}$ and one equips $\mathbb{D}_B$ with the norm

$$\| U \|_{\mathbb{D}_B} = \left(\| U \|_{L^2(m;B)}^2 + \frac{1}{2} \| U^{\#} \|_{L^2(m \times \tilde{m};B)}^2 \right)^{\frac{1}{2}}.$$

Example : If B is a separable Hilbert space we have the following characterization, for $U \in L^2(m; B)$

$$U \in \mathbb{D}_B$$

$$\Updownarrow$$

There is an orthonormal basis $(e_i)_{i\in\mathbb{N}}$ of B such that
$$\forall i \in \mathbb{N}, < e_i, U > \in \mathbb{D} \text{ and } \sum_{i=0}^{\infty} \mathcal{E}[< e_i, U >] < \infty.$$

and in this case

$$U^{\#} = \sum_{i=0}^{\infty} < e_i, U >^{\#} e_i.$$

2) Extension of the functional calculus and image of an error structure by an element of its vectorial domain

• Functional Calculus

Proposition : Let F be a Lipschitz function from B into $\mathbb{R}$.
a) If $U \in \mathbb{D}_B$, $F(U) \in \mathbb{D}$ and $\Gamma[F(U)] \leq K^2 \mathbb{E}_{\tilde{m}}[\| U^{\#} \|_B^2]$.
b) Moreover, if we suppose that F is of class C^1,

$$\Gamma[F(U)] = \mathbb{E}_{\tilde{m}}[< F'(U), U^{\#} >^2]. \quad (*)$$

Remark : If $B = \mathbb{R}^p$

$$(*) \Leftrightarrow \Gamma[F(U)] = \sum_{i,j=1}^{p} F_i'(U) F_j'(U) \Gamma[U_i, U_j]$$

- **Image**

Now, the $\mathbb{D}$-law of $U \in \mathbb{D}_B$ may be defined naturally because the well-defined bilinear form

$$(C^1(B,\mathbb{R}) \cap Lip, F \mapsto \mathcal{E}[F(U)])$$

is closable and its smallest closed extension $(\mathbb{D}_U, \mathcal{E}_U)$ is a local Dirichlet form with a squared field operator denoted by Γ_U. We set

$$U_* S = (B, \mathcal{B}(B), U_* m, \mathbb{D}_U, \Gamma_U).$$

- **$\mathbb{D}$-independence**

The random variables U_1 and $U_2 \in \mathbb{D}_B$ are said to be $\mathbb{D}$-independent if

$$(U_1)_* S \otimes (U_2)_* S = (U_1, U_2)_* S.$$

- **Convergence in $\mathbb{D}$-law**

We say that a sequence $(U_n)_{n \in \mathbb{N}} \in \mathbb{D}_B$ converges in $\mathbb{D}$-law if there exists an error structure $\tilde{S} = (B, \mathcal{B}(B), P, \tilde{\mathbb{D}}, \tilde{\Gamma})$ such that :
i) $(U_n)_* m \to P$ weakly in B,
ii) $C^1(B,\mathbb{R}) \cap Lip \subset \tilde{\mathbb{D}}$ and $\forall F \in C^1(B,\mathbb{R}) \cap Lip$,
$\mathcal{E}[F(U_n)] \to \tilde{\mathcal{E}}[F]$.

Remark : The convergence in $\mathbb{D}_B$ implies the convergence in $\mathbb{D}$-law.

3) Extension of the CLT in separable Hilbert spaces

Let $(H, <>)$ be a separable Hilbert space and $(U_n)_{n\in\mathbb{N}^*}$ be a sequence of centred variables of $\mathbb{D}_H$, Dirichlet-independent with the same Dirichlet law. If we denote by Σ the covariance operator of U_1, then, $V_n = \frac{U_1+\ldots+U_n}{\sqrt{n}}$ converges in Dirichlet law toward $\tilde{S} = (H, \mathcal{B}(H), \nu, \tilde{\mathbb{D}}, \tilde{\Gamma})$ where

i) ν is a centred gaussian measure on H with Σ as covariance operator,

ii) $\forall F \in C^1(H, \mathbb{R}) \cap Lip$, $F \in \tilde{\mathbb{D}}$ and

$$\tilde{\mathcal{E}}[F] = \frac{1}{2} \int_{H^2} < F'(x), y >^2 \, d\mu(x, y)$$

where μ is a centred gaussian measure on H^2 with a covariance operator K defined by $\forall z = (z_1, z_2), \tilde{z} = (\tilde{z}_1, \tilde{z}_2)$ in H^2

$$< Kz, \tilde{z} >_{H^2} = < \Sigma z_1, \tilde{z}_1 > + \mathcal{E}[< U_1, z_2 >, < U_1, \tilde{z}_2 >],$$

iii) $(C^1(H, \mathbb{R}) \cap Lip, \tilde{\mathcal{E}})$ is closable. Its smallest closed extension is $(\tilde{\mathbb{D}}, \tilde{\mathcal{E}})$ that admits a squared field operator $\tilde{\Gamma}$.

Thus (V_n) converges in $\mathbb{D}$-law toward an error structure of **the Ornstein-Uhlenbeck type.**

In fact if $F = f(< e_{i_1}, . >, ..., < e_{i_p}, . >)$,

$$\tilde{\Gamma}[F] = \sum_{k,l=1}^{p} f'_k f'_l (< e_{i_1}, x >, ..., < e_{i_p}, x >) a_{i_k, i_l}$$

where $a_{i_k,i_l} = \mathcal{E}[< e_{i_k}, U_1 >, < e_{i_l}, U_1 >]$.

Sketch of the proof : i) Classical CLT.

ii) By the functional calculus on $\mathbb{D}_H$

$$\mathcal{E}[F(V_n)] = \frac{1}{2} \int_{W \times \tilde{W}} < F'(V_n), V_n^{\#} >_H^2 \, dm \, d\tilde{m}$$

$$= \frac{1}{2} \int_{W \times \tilde{W}} \Phi(V_n, V_n^{\#}) dm \, d\tilde{m}$$

where is a continuous function such that

$$|\Phi(x, y)| \leq K \|(x, y)\|_{H^2}^2.$$

Moreover we can restrict ourselves to the case where $(U_n, U_n^{\#})$ is a sequence of $i.i.d$ random variables in H^2. By the classical CLT in H^2, $(V_n, V_n^{\#})$ converges weakly to a gaussian measure μ with zero mean and a covariance operator equals to K. Finally since

$$\mathbb{E}_P \mathbb{E}_{\tilde{P}} [\| V_n \|_H^2 + \| V_n^{\#} \|_H^2] = \mathbb{E}_m [\| U_1 \|_H^2] + \mathbb{E}_m \mathbb{E}_{\tilde{m}} [\, \| U_1^{\#} \|_H^2]$$

$$= \int_{H^2} \| x \|_{H^2}^2 \, d\mu(x),$$

the weak convergence of $(V_n, V_n^{\#})$ may be extended to sub-quadratic functions thus,

$$\mathcal{E}[F(V_n)] \underset{n\to\infty}{\longrightarrow} \frac{1}{2}\int_{H^2} < F'(x), y >^2 \, d\mu(x,y) = \tilde{\mathcal{E}}[F].$$

iii) From $U_1 \in \mathbb{D}_H$,

$$T : \left(\begin{array}{c} H^2 \to \mathbb{R} \\ (x,y) \mapsto \mathcal{E}[< x, U_1 >, < y, U_1 >] \end{array} \right)$$

is continuous because

$$\mathcal{E}[< x, U_1 >, < y, U_1 >] \leq \| x \|_H \| y \|_H \, \mathbb{E}_{\tilde{m}}\mathbb{E}_m[\| U^{\#} \|_H^2].$$

Therefore, there exists a bounded operator $C : H \to H$ such that $T(x,y) =< Cx, y >$. Setting

$$D_C = C^{\frac{1}{2}} D$$

where D stands for the Fréchet derivative in H we have the following equality : $\forall F \in C^1(H, \mathbb{R}) \cap Lip$

$$\tilde{\mathcal{E}}[F] = \int_H \| D_C[F] \|_H^2 \, d\nu.$$

But using the same argument than one used in a recent paper by Goldys and Gozzi we show that

$$(C^1 \cap Lip, D_C) \text{ is closable in } L^2(\nu, H)$$

$$\Updownarrow$$

$$(\Sigma^{\frac{1}{2}}(H), C^{\frac{1}{2}}\Sigma^{-\frac{1}{2}}) \text{ is closable in } H.$$

Moreover, $(\Sigma^{\frac{1}{2}}(H), C^{\frac{1}{2}}\Sigma^{-\frac{1}{2}})$ is closable because $\mathcal{E}$ is a closed form. $\square$

Remark : The extension of this theorem for separable Banach spaces of type 2 is an open question.

APPROXIMATIONS OF THE B.M

1) Framework

Let us denote by $(\mathcal{C} = C_0([0,1],\mathbb{R}), \|\|_\infty)$ the Wiener space equipped with the Wiener measure ν.
In this section we are interested in the following approximations of the B.M :

$$Y_n(t) = \sum_{k=1}^{n} g_k \int_0^t \phi_k(s)ds \quad \forall t \in [0,1] \tag{2}$$

where $(\phi_k)_{k\in\mathbb{N}^*}$ is an orthonormal basis of $L^2([0,1],dx)$ and $(g_k)_{k\in\mathbb{N}^*}$ a sequence of i.i.d gaussian random variables (with zero mean and variance equals to 1) defined on the probability space $(W,\mathcal{W},P)$.

Remark : 1) When $(\phi_k)_{k\in\mathbb{N}^*}$ is the trigonometric basis of $L^2([0,1],dx)$, (Y_n) is the sequence studied by Wiener in his historical construction of the B.M (1923).
2) When $(\phi_k)_{k\in\mathbb{N}^*}$ is the Haar basis, (Y_n) is the Lévy polygonal approximation of the B.M.

We have the following classical convergence result :

Theorem 1 : The sequence (Y_n) converges uniformly almost-surely and in $L^2(P;\mathcal{C})$ toward a stochastic process having the same law than the B.M.

There are three ways to prove this result :

- **Garsia, Rodemich, Rumsey approach**.
- **The vectorial martingale theorem**.
- **The Lévy-Îto-Nisio theorem**.

2) Extensions

a) **First extension**

We suppose that the (g_k) are erroneous and that their errors are i.i.d and modelised by the Ornstein-Uhlenbeck structure on $\mathbb{R}$. In other words the (g_k) are the coordinate mappings of the following error structure

$S = (W, \mathcal{W}, P, \mathbb{D}, \Gamma) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), m = \mathcal{N}(0,1), H^1(m), (u')^2)^{\mathbb{N}^*}$.

It is easy to construct a sharp operator on S thanks to a copy $(\tilde{W}, \tilde{\mathcal{W}}, \tilde{P})$ of $(W, \mathcal{W}, P)$ putting, for
$U = F(g_1, .., g_n, ..) \in \mathbb{D}$

$$U^{\#} = \sum_{i=1}^{\infty} \frac{\partial F}{\partial g_i}(g_1, \ldots, g_n, \ldots)\tilde{g}_i$$

where the $(\tilde{g}_k)$ are the coordinate mappings of the copy space.

We can see that $\forall n \in \mathbb{N}^*$ $Y_n \in \mathbb{D}_{\mathcal{C}}$ and we have the expected result :

Proposition 1 $(Y_n(.))$ converges in $\mathbb{D}$-law toward the Ornstein-Uhlenbeck structure on the Wiener space.

Proof : It is an immediate consequence of the functional calculus in $\mathbb{D}_{\mathcal{C}}$ and of the following equality

$$Y_n^{\#}(t) = \sum_{k=1}^{n} \tilde{g_k} \int_0^t \phi_k(s)ds.\square$$

Corollary 1 : The sequence (Y_n) converges uniformly quasi surely according to the capacity associated to the error structure S.

Corollary 2 : $\|Y_n(.)\|_\infty$ (resp. $\underset{t\in[0,1]}{sup}\ X_n(t)$) converges in $\mathbb{D}$-law toward $(\|B.\|_\infty)_* S_{OU}$ (resp. $\underset{t\in[0,1]}{sup}\ (B_t)_* S_{OU}$).

Remark : This method can be generalized to any gaussian process that satisfies the Fernique continuity criterium. In particular this is the case of the fractional B.M with Hurst parameter $0 < H < 1$.

b) **Interpretation in terms of simulation**

Suppose we want to simulate the B.M using the random serie

$$Y_n(t) = \sum_{k=1}^{n} g_k \int_0^t \phi_k(s)ds \quad \forall t \in [0,1] \tag{3}$$

(this is known as the Karhunen-Loève method). Thus we have to generate a sequence of i.i.d gaussian variables. If we use the inversion method, it suffices to generate an i.i.d family (U_k) of uniform random variables and to set $g_k = F^{-1}(U_k)$.

The preceding proposition tell us that if we suppose that the errors on the U_k are independent (even if this random variables are generated with the same generator) and modelised by the pseudo-gaussian error structure on $[0,1]$, the error on the simulated B.M can be represented by the Ornstein-Uhlenbeck structure on the Wiener space.

In fact the pseudo gaussian structure is defined as

$$F_*(\mathbb{R}, \mathcal{B}(\mathbb{R}), m = \mathcal{N}(0,1), H^1(m), (u')^2).$$

This leads us to the natural question : What happens when we modelised the error on U_k by another error structure ?

c) **Second extension**

Let $a : [0,1] \to \mathbb{R}$ be a non negative continuous function such that

$$S = ([0,1], \mathcal{B}([0,1]), dx, \mathbb{d}, f \mapsto a(x)(f'(x))^2)$$

is an error structure with $C^1([0,1]) \subset \mathbb{d}$ (this imposes conditions on a in relation with the Hamza theorem). If we suppose that

$$c = \int_0^1 \frac{a}{[F'(F^{-1})]^2} dx < \infty \quad (*)$$

we can show that $F^{-1} \in \mathbb{d}$. Thus we have the following result :

Proposition 2 : Let us suppose that the U_k are the coordinate mappings of the error structure S^{N^*} and that the condition $(*)$ holds, then (Y_n) converges in Dirichlet law toward S^c_{OU}.

d) **Third extension**

The preceding results remain valid if the gaussian variables g_k are generated by other methods (Box Muller, CLT).