ERROR CALCULUS : the language of DIRICHLET FORMS.

STATISTICAL IDENTIFICATION (dim ∞)

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STUDY PLAN

Error calculus : the language of Dirichlet forms

- Intuitive notion of error structure
- An extension tool
- Image and infinite products of error structures : The Ornstein-Uhlenbeck structure on Wiener space
- The Black-Scholes model (I)

Statistical identification

- D-independence, convergence in D-law
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Intuitive notion of error structures

Let us consider a quantity C able to be measured by an experimental device which result exhibits an error denoted by $\triangle C$. We will consider that those quantities are represented by random variables generally correlated .

WE ARE INTERESTED IN ERROR PROPAGATION

Probabilistic approach :

It is required to know the joint law of the pair $(C, \triangle C)$ to model the experiment. Thus the study of error transmission is associated to the calculus of images of probability measures. Unfortunately, the knowledge of the law of $\triangle C$ given C by means of experiment is practically impossible. For pragmatic purposes, we adopt the following assumptions :

Intermediate approach :

$\mathbf{H}_1$ **:** We will suppose that the conditional variance $var[\triangle C \mid C]$ is known and that $\mathbb{E}[\triangle C \mid C] = 0$.

$\mathbf{H}_2$ **:** We assume that the errors are small to allow the simplication usually performed by engineers and to use classical differential calculus. Thus $\triangle C = \varepsilon Y$ where Y is a bounded random variable and ε a size parameter.

If f, g are in $C^3(\mathbb{R}, \mathbb{R})$ with bounded derivatives, using Taylor's formula it follows that :

1)

$$var[\triangle f(C) \mid C] = f'^2(C) var[\triangle C \mid C] + \varepsilon^3\ 0(1)$$

$$\mathbb{E}[\triangle f(C) \mid C] = \tfrac{1}{2} f''(C) var[\triangle f(C) \mid C] + \varepsilon^3\ 0(1).$$

2)

$$var[\triangle(g \circ f(C)) \mid C] = g'^2(f(C)) var[\triangle f(C) \mid C] + \varepsilon^3\ 0(1)$$

$$\mathbb{E}[\triangle(g \circ f(C)) \mid C] = g'(f(C)) \mathbb{E}[\triangle f(C) \mid C] + \frac{1}{2} g''(f(C)) var[\triangle f(C) \mid C] + \varepsilon^3\ 0(1).$$

We can see that the calculus on the variances is a first order calculus and does not involve the biases when the calculus on the biases is of the second order and involves the variances.

On the space $(\mathbb{R}, Bor(\mathbb{R}), law\ of\ C)$, we introduce the operator Γ^C, called the quadratic error operator, defined by

$$\Gamma^C[f](x) = \underset{\varepsilon \to 0}{Lim} \frac{var[\triangle f(C) \mid C = x]}{\varepsilon^2}.$$

Γ^C naturally polarizes into a bilinear, symmetric, positive operator fulfilling the following functional calculus : If $F \in C^2(\mathbb{R}^2)$ with bounded partial derivatives

$$\begin{aligned}\Gamma^C[F(f,g)] = F_1'^2(f,g)\Gamma^C[f] + F_2'^2(f,g)\Gamma^C[g] \\ +2F_1'(f,g)F_2'(f,g)\Gamma^C[f,g].\end{aligned}$$

Propagation "à la Gauss" (1821)

The term $(\mathbb{R}, Bor(\mathbb{R}), loi\ de\ C, \Gamma^C)$ is called an error structure.

AN EXTENSION TOOL

From now on, an error structure is a term $(W, \mathcal{W}, m, \mathbb{D}, \Gamma)$ where $(W, \mathcal{W}, m)$ is a probability space, $\mathbb{D}$ is a dense vector subspace of $L^2(m)$ and Γ is a positive symmetric bilinear map from $\mathbb{D} \times \mathbb{D}$ into $L^1(m)$ fulfilling :

1) the functional calculus of class $C^1 \cap Lip$ i.e. if $U = (U_1, \ldots, U_n) \in \mathbb{D}^n$, $F \in C^1(\mathbb{R}^n) \cap Lip$ then $F(U_1, \ldots, U_n) \in \mathbb{D}$ and

$$\Gamma[F(U_1, \ldots, U_n)] = \sum_{i,j} F_i'(U) F_j'(U) \Gamma[U_i, U_j],$$

2) $1 \in \mathbb{D}$ (this implies that $\Gamma[1] = 0$),

3) the bilinear form $\mathcal{E}[F, G] = \frac{1}{2} \int \Gamma[F, G] dm$ defined on $\mathbb{D} \times \mathbb{D}$ is closed i.e. $\mathbb{D}$ is complete under the norm

$$\| \,.\, \|_{\mathcal{E}} = (\| \,.\, \|^2_{L^2(m)} + \mathcal{E}[.])^{\frac{1}{2}}.$$

We always write $\Gamma[F]$ for $\Gamma[F, F]$ and $\mathcal{E}[F]$ for $\mathcal{E}[F, F]$.

From the hypotheses mentioned above, $\mathcal{E}$ is a local Dirichlet form and Γ its associated squared field operator.

COMMENTS

• Natural extension of the intuitive notion (property 1)
• Image and products are quite natural
• Property (3) ensures that the domain $\mathbb{D}$ is preserved by Lipschitz functions : if F is a contraction then for $U \in \mathbb{D}$ one has $F(U) \in \mathbb{D}$ and

$$\Gamma[F(U)] \leq \Gamma[U].$$

• Without other hypotheses we have a calculus on biases.

EXAMPLE

Ornstein-Uhlenbeck structure in dimension 1

$$(\mathbb{R}, \mathcal{B}(\mathbb{R}), m = \mathcal{N}(0,1), H^1(m), \Gamma[u] = u'^2)$$

Images and Infinite Products

IMAGE

Let $S = (W, \mathcal{W}, m, \mathbb{D}, \Gamma)$ be an error structure and $U : (W, \mathcal{W}) \to (E, \mathcal{B}(E))$ a random variable with values in a general metric space E.

We want to define the image of S by U.

We consider a set $\mathcal{A}$ of maps from E into $\mathbb{R}$ such that :

i) $\mathcal{A}$ is a dense vector subspace of $L^2(U_* m)$,
ii) $\forall\, G \in C_K^\infty(\mathbb{R})$ and $F \in \mathcal{A}$ then $G(F) \in \mathcal{A}$,
iii) $\forall F \in \mathcal{A}$, $F(U) \in \mathbb{D}$.

The bilinear form $\mathcal{E}_{\mathcal{A},U}$ defined on $\mathcal{A}$ by $\underline{\mathcal{E}_{\mathcal{A},U}[F] = \mathcal{E}[F(U)]}$ can be extended to a domain $\mathbb{D}_{\mathcal{A},U}$ such that $(\mathbb{D}_{\mathcal{A},U}, \mathcal{E}_{\mathcal{A},U})$ is a Dirichlet form with a squared field operator denoted by $\Gamma_{\mathcal{A},U}$. The error structure

$$U_*^{(\mathcal{A})} S = (E, \mathcal{B}(E), U_* m, \mathbb{D}_{\mathcal{A},U}, \Gamma_{\mathcal{A},U})$$

is the image of S by U with respect to $\mathcal{A}$.

It is called the **D-law of** U **with respect to** $\mathcal{A}$.

Remark : In general such a definition depends on the choice of $\mathcal{A}$.

When $(E, <>)$ is a separable Hilbert space, additional hypotheses on U make this definition be canonical : we set

$$\mathcal{A} = \{F = f(< x_1, . >, \ldots, < x_p, . >; p \in \mathbb{N}^*,$$

$$(x_1, \ldots x_p) \in E^p, f \in C^1 \cap Lip(\mathbb{R}^p)\},$$

and for an orthonormal basis $\beta = (e_i)_{i \in \mathbb{N}}$ we define

$$\mathcal{A}_\beta = \{F = f(< e_1, . >, \ldots, < e_p, . >; p \in \mathbb{N}^*,$$

$$(e_1, \ldots e_p) \in \beta, f \in C^1 \cap Lip(\mathbb{R}^p)\}.$$

If $U \in \mathbb{D}_H$ (vectorial domain of a dirichlet form in the sense of Feyel, de La Pradelle) we have

$$U_*^{(\mathcal{A})} S = U_*^{(\mathcal{A}_\beta)} S.$$

Infinite products

Let $S_n = (W_n, \mathcal{W}_n, m_n, \mathbb{D}_n, \Gamma_n)$, $n \geq 0$, be a familly of error structures. The product structure

$$(W, \mathcal{W}, m, \mathbb{D}, \Gamma) = \prod_{n=0}^{\infty} S_n$$

is defined by $(W, \mathcal{W}, m) = (\prod_{n=0}^{\infty} W_n, \prod_{n=0}^{\infty} \mathcal{W}_n, \otimes_{n=0}^{\infty} m_n)$ with an explicit domain $\mathbb{D}$ and $\forall F \in \mathbb{D}$,

$$\Gamma[F] = \sum_{n=0}^{\infty} \Gamma_n[F]$$

where Γ_n acts on the n-th argument of F.

From the error structures in dimension 1, it is easy to provide, using the preceding operations, error structures on usual spaces of stochastic proccesses e.g on Wiener space

Example : The Ornstein-Ulhenbeck structure on Wiener space.

• We set $B = \{f : [0,1] \to \mathbb{R} | f \ continuous \ and \ f(0) = 0\}$, B' its topological dual space, and ν the Wiener measure.

We consider a product error structure

$$S = \prod_{n \in \mathbb{N}^*} S_n$$

where $\forall n \in \mathbb{N}^*$,

$$S_n = (\mathbb{R}, \mathcal{B}(\mathbb{R}), m = \mathcal{N}(0,1), H^1(m), \Gamma[u] = u'^2).$$

• We define

$$Z^0 : (t \to \sum_{n \in \mathbb{N}^*} \int_0^t \phi_n(s) ds . g_n)$$

with the $g_n's$ being the coordinate mappings of S and where $(\phi_k)_{k \in \mathbb{N}^*}$ is an orthonormal basis of $L^2([0,1], dx) = H$

We set

$$\mathcal{A} = \{F(\lambda_1, \ldots, \lambda_n); n \in \mathbb{N}, (\lambda_1, \ldots, \lambda_n) \in B' \ et \ F \in C^1 \cap Lip\}.$$

We denote by

$$S^0 = (B, \mathcal{B}(B), \nu, \mathbb{D}_0, \Gamma_0)$$

the image of S by Z^0 with respect to $\mathcal{A}$. This is the **Ornstein-Uhlenbeck structure** on B.

We can see that for $f = F(\lambda_1, \ldots, \lambda_n) \in \mathcal{A}$:

$$\Gamma_0[f] = \sum_{i,j=1}^{n} \partial_i F \partial_j F(\lambda_1, \ldots, \lambda_n) a_{i,j}$$

where the coefficients $a_{i,j}$ are independent of F.

This structure is very important, it is a natural and simple way to introduce the so-called Malliavin calculus : there exists an operator $\nabla : \mathbb{D}_0 \to L^2(\nu; H)$ such that

$$\Gamma_0[F] = \| \nabla[F] \|^2_{L^2([0,1],dx)} .$$

∇ is none other than the gradient in the Malliavin sense with an adjoint that extend the Ito integral.

THE BLACK-SCHOLES MODEL

Let us denote by r the interest rate of the bond. The asset (S_t) is modeled as the solution to the equation

$$dS_t = S_t(rdt + \sigma dB_t)$$

in such a way that the discounted stock price is a martingale. For an European option with payoff $f(S_1)$ the value of the simulating portfolio V_t at time t is

$$V_t = F(t, S_t, \sigma, r)$$

where

$$F(t,x,\sigma,r) = e^{-r(T-t)} \int f\left(xe^{(r-\frac{\sigma^2}{2})(T-t)+\sigma y\sqrt{T-t}}\right) \frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}} dy.$$

If f is borel with linear growth, F is regular and we can define the following quantities :

$$delta_t = \frac{\partial F}{\partial x}(t, S_t, \sigma, r)$$
$$gamma_t = \frac{\partial^2 F}{\partial x^2}(t, S_t, \sigma, r).$$

There are two parameters in B & Smodel :

-the volatility σ (finite dimensional parameter)

-The brownian motion $(B_t)_{t\in[0,1]}$ defined on the Wiener space (functional parameter).

By statistical methods, we want to associate to σ and to the B.M error structures that will express the precision of our knowledge on this parameters.

• For σ, an historical observation of stock prices provides naturally an error structure via the concept of **FISHER INFORMATION**.

Cf *N.Bouleau et Ch.Chorro, "Error structure and parameter estimation", note C.R.A.S, February 2004.*

• For the **B.M** the preceding method is meaningless. As the classical C.L.T explains the importance of the normal law in practical applications we are going to prove convergence theorems that give a natural structure on the Wiener space.

D-independence, Convergence in D-law

In this section $S = (W, \mathcal{W}, m, \mathbb{D}, \Gamma)$ is fixed.

Here we extend the notion of independence and convergence in distribution to error structures.

Let $(U_n)_{n\in\mathbb{N}}$ be random variables from W into E and $\mathcal{A}$ a set such that $(U_n)_*^{(\mathcal{A})}S$ is well defined.

D-independence

U_1 and U_2 are said to be D-independent if

$$(U_1) *^{(\mathcal{A})} S \otimes (U_2) *^{(\mathcal{A})} S = (U_1, U_2)_*^{(\mathcal{A})} S.$$

Convergence in D-law

We say that $(U_n)_{n\in\mathbb{N}}$ converge in D-law with respect to $\mathcal{A}$ if there exists an error structure $\tilde{S} = (E, \mathcal{B}(E), p, \tilde{\mathbb{D}}, \tilde{\Gamma})$ such that :

i) $(U_n)_* m \to p$ in distribution in E,

ii) $\mathcal{A} \subset \tilde{\mathbb{D}}$ and $\forall F \in \mathcal{B}$, $\mathcal{E}[F(U_n)] \to \tilde{\mathcal{E}}[F]$.

Convergence theorem

A central limit theorem for hilbert-valued random variables

Let us suppose that (U_n) is a sequence in $\mathbb{D}_H$ of Dirichlet-independent random variables with the same Dirichlet-law. If we suppose that U_1 has zero mean and a covariance operator denoted by Σ then, $V_n = \frac{U_1+\ldots+U_n}{\sqrt{n}}$ converge in Dirichlet-law with respect to $\mathcal{A}$ toward $\tilde{S} = (H, \mathcal{B}(H), \nu, \tilde{\mathbb{D}}, \tilde{\Gamma})$ where

i) ν is a gaussian measure on H with zero mean and covariance operator Σ,

ii) $\forall F \in \mathcal{A}$, $F = f(< x_1, . >, \ldots, < x_p, . >) \in \tilde{\mathbb{D}}$ and

$$\tilde{\Gamma}[F] = \sum_{i,j=1}^{p} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}(< x_1, . >, \ldots, < x_p, . >)a_{i,j}$$

where $a_{i,j} = \mathcal{E}[< x_i, U_1 >, < x_j, U_1 >]$,

iii) $(\mathcal{A}, \tilde{\mathcal{E}})$ is closable and we denote by $(\tilde{\mathbb{D}}, \tilde{\mathcal{E}})$ its smallest closed extension.

An extension of the Donsker's theorem

Here we suppose that (U_n) is a sequence of random variables in $\mathbb{D}$, Dirichlet-independent with the same Dirichlet-law. For $t \in [0,1]$, $n \in \mathbb{N}$ we put

$$X_n(t) = \frac{1}{\sqrt{n}}(\sum_{k=1}^{[nt]} U_k + (nt - [nt])U_{[nt]+1})$$

where $[nt]$ is the interger part of nt.

According to Donsker's theorem the process X_n converges in distribution toward ν in B.

If we put $\pi_t : f \in B \to f(t)$ and

$$\mathcal{A} = \{F(\pi_{t_1}, \ldots, \pi_{t_n}); n \in \mathbb{N}, (t_1, \ldots, t_n) \in [0,1]^n$$

$$and\ F \in C^1(\mathbb{R}^n) \cap Lip\},$$

we have the following extension

X_n **converges in Dirichlet-law with respect to** $\mathcal{A}$ **toward the Ornstein-Uhlenbeck structure on B.**

Black and Scholes (II)

If we simulate the brownian motion by means of random walks, the natural error structure expressing the precision of the method is the Ornstein-Ulhenbeck structure on B.

We obtain the following results :

$$\Gamma_0[S_t, S_s] = S_s S_t (s \wedge t)$$

$$\Gamma_0[V_s, V_t] = delta_s delta_t \Gamma_0[S_s, S_t]$$

$$\Gamma_0[H_t, H_s] = gamma_s gamma_t \Gamma_0[S_s, S_t].$$

- Extension of the calculus of Greeks.
- Error on $\int_0^1 e^{-rs} H_s S_s ds$
- General diffusion models.