On an extension of the Hilbertian Central Limit Theorem to Dirichlet Forms

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Abstract: In a recent paper ([2]), Nicolas Bouleau provides a new tool, based on the language of Dirichlet forms, to study the propagation of errors and reinforce the historical approach of Gauss. In the same way that the practical use of the normal distribution in statistics may be explained by the central limit theorem, the aim of this paper is to underline the importance of a family of error structures by asymptotic arguments.

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1. Introduction

The choice of a relevant mathematical language for speaking about errors and their propagations is an old topic. A new approach based on the theory of Dirichlet forms ([1],[9],[14]) has been recently suggested in [2],[3]. This method is a natural and powerful extension of the pioneer works of Gauss ([2]) and it seems to be an appropriate framework to study the sensitivity to small changes of parameters in physical and financial models ([3], chap.7).

From now on, we shall use the expression ‘error structure’ to denote a term \((W,W,m,D,\Gamma)\) where \((W,W,m)\) is a probability space, \(D\) is a dense sub-vector space of \(L^2(W,\mathcal{W},m)\) (also denoted by \(L^2(m)\)) and \(\Gamma\) is a positive symmetric bilinear map from \(D \times D\) into \(L^1(m)\) fulfilling:
1) the functional calculus of class $C^1 \cap \text{Lip}$ meaning that if $U \in \mathbb{D}^n$, $V \in \mathbb{D}^p$, for $F \in C^1(\mathbb{R}^n, \mathbb{R}) \cap \text{Lip}$ = \{C^1 \text{ and Lipschitz}\} and $G \in C^1(\mathbb{R}^p, \mathbb{R}) \cap \text{Lip}$ one has $(F(U), G(V)) \in \mathbb{D}^2$ and

$$\Gamma[F(U), G(V)] = \sum_{i,j} \frac{\partial F}{\partial x_i}(U) \frac{\partial G}{\partial x_j}(V) \Gamma[U_i, V_j] \text{ m-a.e},$$

2) $1 \in \mathbb{D}$ (this implies $\Gamma[1, 1] = 0$),

3) the bilinear form $\mathcal{E}[F, G] = \frac{1}{2} \mathbb{E}_m[\Gamma[F, G]]$ defined on $\mathbb{D} \times \mathbb{D}$ is closed i.e. $\mathbb{D}$ is complete under the norm of the graph

$$\| \cdot \|_\mathbb{D} = (\| \cdot \|_{L^2(m)}^2 + \mathcal{E}[\cdot])^{\frac{1}{2}}.$$

We always write $\Gamma[F]$ for $\Gamma[F, F]$ and $\mathcal{E}[F]$ for $\mathcal{E}[F, F]$.

From the hypotheses mentioned above, $\mathcal{E}$ is a local Dirichlet form and $\Gamma$ its associated squared field operator. The property 1) is none other than the so-called Gauss’ law of small errors propagation ([2]), thus, when $U \in \mathbb{D}$, the intuitive meaning of $\Gamma[U]$ is the conditional variance of the error on $U$ given $U$. Moreover this first order calculus dealing with variances can naturally be reinforced by a calculus on biases involving the infinitesimal generator associated to $\mathcal{E}$ ([3], chap.3).

Thanks to property 3), the domain $\mathbb{D}$ is preserved by Lipschitz functions ([3], p.40): if $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is a contraction in the following sense

$$|F(x) - F(y)| \leq \sum_{i=1}^n |x_i - y_i|$$

then for $U = (U_1, \ldots, U_n) \in \mathbb{D}^n$ one has $F(U) \in \mathbb{D}$ and

$$\Gamma[F(U_1, \ldots, U_n)]^{\frac{1}{2}} \leq \sum_{i=1}^n \Gamma[U_i]^{\frac{1}{2}}.$$

As mentioned in [4], one of the lacks of this new theory in practical cases is the necessity of a priori choices. In fact, for a rational treatment, error hypotheses should be obtained by statistical methods. In finite dimension, error structures are connected (through a robust identification) to statistical parametric methods by means of Fisher information [4]. Moreover, this study can be reinforced by the refinement of the main limit theorems of the probability theory in our setting ([5],[6]).

In this way, Bouleau and Hirsch have introduced notions of independence and convergence for error structures that extend the independence and the convergence in distribution for random variables ([1], chap.5). By using these definitions, they prove a central limit theorem in finite dimension for erroneous random variables, the errors being modelised by error structures ([1],
The main contribution of our paper is to propose an infinite dimensional extension of this result, at the very least, in the case of a separable Hilbert space. This finding, associated with the recent improvements of the Donsker theorem ([5],[6]), can explain the importance of the error structures of the Ornstein-Uhlenbeck type (structures where the measure is Gaussian and where $\Gamma$ operates on cylinder functions as a first order differential operator with constant coefficients) in the applications.

From a technical point of view, the key stone of our study will be the notion of the vectorial domain of a Dirichlet form which is due to Feyel and de La Pradelle ([8], p.900).

2. Preliminaries on error structures

2.1. Finite dimensional images and infinite products.

Let us present the two fundamental algebraic operations on error structures that are compatible with the construction of probability spaces. We refer to [4] for their statistical interpretations.

**Definition 1.** Let $S = (W, \mathcal{W}, m, D, \Gamma)$ be an error structure and $U$ a random variable in $D^d$. For $f \in C^1(\mathbb{R}^d, \mathbb{R}) \cap \text{Lip}$, we put
\[
\Gamma_U[f](x) = \mathbb{E}_m[\Gamma[f(U)] | U = x].
\]

Thus, $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), U_* m, C^1(\mathbb{R}^d, \mathbb{R}) \cap \text{Lip}, \Gamma_U)$ is a closable error pre-structure in the sense of [3], p.44. We denote by $U_* S$ its smallest closed extension called the image structure of $S$ by $U$.

**Definition 2.** Let $S_n = (W_n, \mathcal{W}_n, m_n, D_n, \Gamma_n)$, $n \geq 0$, be a family of error structures. The product structure $(W, \mathcal{W}, m, D, \Gamma) = \prod_{n=0}^{\infty} S_n$ is defined by $(W, \mathcal{W}, m) = (\bigotimes_{n=0}^{\infty} W_n, \bigotimes_{n=0}^{\infty} \mathcal{W}_n, \otimes_{n=0}^{\infty} m_n)$ with an explicit domain $D$ ([1],p.203) and $\forall F \in D, \Gamma[F] = \sum_{n=0}^{\infty} \Gamma_n[F]$ where the operator $\Gamma_n$ acts on the n-th variable.

Thanks to the preceding definitions, it is easy to equip with error structures the fundamental spaces encountered in stochastic models (Wiener space, Monte Carlo space, Poisson space) starting from elementary structures on $\mathbb{R}$ ([3], Chap.6).

Now, in order to deal with Hilbert valued random variables, we have first to give sense to a coherent extension of the domain of an error structure.
2.2. Vectorial domain of an error structure.

From now on, we suppose that the error structure $S$ satisfies the property (G) defined as follows. In practice, this assumption is not restrictive because Mokobodski showed a gradient exists whenever $\mathbb{D}$ is separable ([1], p.242).

**Definition 3.** We say that an error structure $S$ owns a gradient if there exists a separable Hilbert space $(\mathcal{H}, \| \cdot \|_{\mathcal{H}})$ and an operator $\nabla$ from $\mathbb{D}$ into $L^2(m; \mathcal{H})$ (where $L^2(m; \mathcal{H})$ denotes the space of square integrable random variables with values in $\mathcal{H}$) called the gradient such that

$$\forall U \in \mathbb{D}, \| \nabla U \|_{\mathcal{H}}^2 = \Gamma[U].$$

Thus, according to relation (1), a gradient fulfills the classical chain rule.

We introduce a slight variant of the gradient which is very useful when computing errors on Wiener space thanks to the Itô formula ([3], p.123). This notion has been introduced by Feyel and de La Pradelle in the Gaussian case and used by Bouleau and Hirsch to prove important results concerning the regularity of solutions of stochastic differential equations with Lipschitz coefficients ([1], chap.4).

**Definition 4.** Let $(\hat{W}, \hat{\mathcal{W}}, \hat{m})$ be a probability space which is a copy of $(W, \mathcal{W}, m)$ and $J$ an isometry from $\mathcal{H}$ into $L^2(\hat{m})$. For $U \in \mathbb{D}$, we denote by $U^\#$ the derivative of $U$ defined by

$$U^\# = J(\nabla U) \in L^2(m \times \hat{m}).$$

We can of course suppose (what we shall do) that $\forall h \in \mathcal{H}, \mathbb{E}_m[J(h)] = 0$.

Let $(B, \| \cdot \|_B)$ be a separable Banach space and $B'$ its topological dual space. We denote by $<,>$ the duality between $B$ and $B'$. One of the main interest of derivative is to allow a definition of a tensor product of $\mathbb{D}$ with $B$.

**Definition 5.** Let us denote by $\mathbb{D}_B$ the vector space of random variables $U$ in $L^2(m; B)$ such that there exists $g$ in $L^2(m \times \hat{m}; B)$ so that

$$\forall \lambda \in B', \ < \lambda, U > \in \mathbb{D} \text{ and } < \lambda, U >^\# = < \lambda, g >.$$

We then put $g = U^\#$ and one equips $\mathbb{D}_B$ with the norm

$$\| U \|_{\mathbb{D}_B} = \left( \| U \|_{L^2(m; B)}^2 + \frac{1}{2} \| U^\# \|_{L^2(m \times \hat{m}; B)}^2 \right)^{\frac{1}{2}}.$$

(Thus $\mathbb{D}_{\mathbb{R}^d} = \mathbb{D}^d$).
Remark 1: Let $B$ be a separable Hilbert. If $U \in L^2(m; B)$ the following statements are equivalent:

i) $U \in \mathcal{D}_B$.

ii) There is an orthonormal basis $(e_i)_{i \in \mathbb{N}}$ of $B$ such that $\forall i \in \mathbb{N}, <e_i, U> \in \mathbb{D}$ and $\sum_{i=0}^{\infty} \mathcal{E}[<e_i, U>] < \infty$.

iii) For all orthonormal basis $(e_i)_{i \in \mathbb{N}}$ of $B$, $\forall i \in \mathbb{N}$, one has $<e_i, U> \in \mathbb{D}$ and $\sum_{i=0}^{\infty} \mathcal{E}[<e_i, U>] < \infty$.

Henceforth, $B$ is assumed to satisfy the approximation hypothesis: There exists a sequence $(P_n)_{n \in \mathbb{N}}$ of continuous linear operators of finite rank from $B$ into $B$ such that $\forall x \in B, \lim_{n \to \infty} P_n(x) = x$ (this assumption holds when $B$ owns a Schauder basis, in particular, when $B$ is a separable Hilbert space or the Wiener space).

The following proposition extends the property (2) of stability of the domain $\mathbb{D}$ by contractions and the functional calculus (1) to $\mathcal{D}_B$.

**Proposition 1.** Let $F$ be a contraction from $B$ into $\mathbb{R}$, if $U \in \mathcal{D}_B$ one has $F(U) \in \mathbb{D}$ and $\Gamma[F(U)] \leq E_m[\|U\|^2_B]$. Moreover, if we suppose that $F$ is of class $C^1$, $F(U)^\# = <F'(U), U^\#>$.

Proof. Let us first suppose that $F \in C^1(B, \mathbb{R}) \cap Lip$. For all $n \in \mathbb{N}$, we define the cylinder approximations of $F$ by $F_n = F \circ P_n$. According to the Banach-Steinhaus theorem one has $\sup_{N \in \mathbb{N}} \|P_N\| < \infty$ thus, by dominated convergence, $F_n(U) \to F(U)$ in $L^2(m)$. Moreover, since $F_n$ is a cylinder function, $F_n(U) \in \mathbb{D}$ and it follows from the chain rule that $F_n(U)^\# = <F'(P_n(U)), P_n(U)^\#>$. Hence, we obtain easily that $F_n(U)^\# \to <F'(U), U^\#>$ in $L^2(m \otimes \hat{m})$. Using the fact that the derivative is a closed operator, the conclusion holds.

When $F$ is only a contraction, the result follows from an adaptation of the proof of theorem 2.2.3 of [1], p.140.

A direct consequence of the preceding proposition is to allow the construction of the image of $S$ by an element of its vectorial domain by using the same idea than in definition 1.

**Definition 6.** Let $U \in \mathcal{D}_B$, the term $(B, \mathcal{B}(B), U_m, C^1(B, \mathbb{R}) \cap Lip, \Gamma_U)$ where $\forall F \in C^1(B, \mathbb{R}) \cap Lip$, $\Gamma_U[F] = E_m[\Gamma[F(U)] | U]$, is a closable error pre-structure in the sense of [1], p.44. Let us denote by $U_* S$ its smallest closed extension, and by $(\mathcal{E}_U, \mathcal{D}_U)$ the associated Dirichlet form. The structure $U_* S$ is called the image of $S$ by $U$ or the Dirichlet law of $U$ and $\forall F \in \mathcal{D}_U$, we have
\[ \mathcal{E}_U[F] = \mathcal{E}[F(U)]. \]

**Example 1.** One of the simplest example of error structure is the term

\[ (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu, H^1(\mu), \gamma : u \mapsto u^2) \]

where \( \mu \) is the reduced normal law on \( \mathbb{R} \) and \( H^1(\mu) \) the first Sobolev space associated to \( \mu \). We consider (definition 2) the following product

\[ S = (W, \mathcal{W}, m, \mathcal{D}, \Gamma) = \prod_{n=0}^{\infty} (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu, H^1(\mu), \gamma). \]

Let us denote by \((g_n)_{n \in \mathbb{N}}\) the coordinate mappings of \( S \) and by \((\phi_n)_{n \in \mathbb{N}}\) an orthonormal basis of \( L^2([0,1], dx) \). For \( t \in [0,1] \), we set

\[ B_t = \sum_{n \in \mathbb{N}} \left( \int_0^t \phi_n(s) ds \right) g_n. \]

Thus, the continuous process \((B_t)_{t \in [0,1]} \) is a standard Brownian motion and we can see easily that it belongs to \( \mathbb{D}_B \) where \( B = C_0([0,1], \mathbb{R}) \) is the Wiener space. The image of \( S \) by \((B_t)_{t \in [0,1]} \) is known as the Ornstein-Uhlenbeck error structure on \( B \) and we denote by \( \Gamma_{OU} \) its squared field operator. This structure possesses a gradient which is none other than the gradient in the Malliavin sense with an adjoint operator that extends the Itô integral ([15]). If we put \( \mathcal{A} = \{ f(\lambda_1, \ldots, \lambda_n); n \in \mathbb{N}, (\lambda_1, \ldots, \lambda_n) \in B' \text{ and } f \in C^1(\mathbb{R}^n, \mathbb{R}) \cap \text{Lip} \} \), we can see that \( \forall F = f(\lambda_1, \ldots, \lambda_n) \in \mathcal{A}, \)

\[ \Gamma_{OU}[F] = \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}(\lambda_1, \ldots, \lambda_n) a_{i,j} \]

where the coefficients \( a_{i,j} \) only depend on \( (\lambda_i, \lambda_j) \). Thus, from now on, we will say that an error structure on a separable Banach space is of the Ornstein-Uhlenbeck type if the associated measure is Gaussian and if the operator \( \Gamma \) is of the form (3) on regular cylinder functions.

Now, we extend the definition of the Dirichlet independence introduced by Bouleau and Hirsch in finite dimension ([1], p.217). The product of two error structures is denoted by \( \otimes \).

**Definition 7.** If \((U, V) \in (\mathbb{D}_B)^2\), \( U \) and \( V \) are said to be Dirichlet independent if \( U_*S \otimes V_*S = (U, V)_*S \). In other terms, the Dirichlet law of \((U, V)\) is the product of the Dirichlet laws of \( U \) and \( V \).
We can show that the theorem 4.1.4, p.218 of [1] remains valid in our framework, thus, we have the following characterization of the Dirichlet independence on $\mathbb{D}_B$.

**Proposition 2.** For $U$ and $V$ in $\mathbb{D}_B$ to be Dirichlet independent, it is necessary and sufficient that the four following conditions are fulfilled:

a) $U$ and $V$ are independent on the probability space $(W, \mathcal{W}, m)$.

b) $\forall (\lambda_1, \lambda_2) \in (B')^2$, $\mathbb{E}_m[\Gamma(\lambda_1(U), \lambda_2(V))|U, V] = 0$ $m$-a.e.

c) $\forall \lambda \in B'$, $\mathbb{E}_m[\Gamma(\lambda(U))|U, V] = \mathbb{E}_m[\Gamma(\lambda(U))|U]$ $m$-a.e.

d) $\forall \lambda \in B'$, $\mathbb{E}_m[\Gamma(\lambda(V))|U, V] = \mathbb{E}_m[\Gamma(\lambda(V))|V]$ $m$-a.e.

Finally, we introduce a notion of convergence on the vectorial domain that reinforce the convergence in distribution for random variables.

**Definition 8.** We say that a sequence $(U_n)_{n \in \mathbb{N}}$ in $\mathbb{D}_B$ converges in Dirichlet law if there exists an error structure $\hat{S} = (B, \mathcal{B}(B), \nu, \hat{D}, \hat{\Gamma})$ such that:

i) $(U_n), m \xrightarrow{n \to \infty} \nu$ weakly

ii) $C^1(B, \mathbb{R}) \cap \text{Lip} \subset \hat{D}$ and $\forall F \in C^1(B, \mathbb{R}) \cap \text{Lip}$, $\mathbb{E}[F(U_n)] \xrightarrow{n \to \infty} \hat{\mathbb{E}}[F]$.

For convenience, we will say that $(U_n)_{n \in \mathbb{N}}$ converges in Dirichlet law toward $\hat{S}$.

In the next section, we prove an extension of the central limit theorem in Hilbert spaces (in the sense of the preceding definition).

### 3. Main result

Suppose that $S = (W, \mathcal{W}, m, \mathbb{D}, \Gamma)$ owns a gradient $\nabla : \mathbb{D} \to L^2(m; \mathcal{H})$. Let us denote by $\# : \mathbb{D} \to L^2(m \times \hat{m})$ a derivative operator. Although noncanonical, the choice of the isometry $J$ is not specified because, according to remark 1, when $H$ is a separable Hilbert space, such a choice leads to the same definition of $\mathbb{D}_H$.

**Theorem 1.** Let $(H, <\cdot, \cdot>)$ be a separable Hilbert space. Let $(U_n)_{n \in \mathbb{N}^*}$ be a sequence of centered random variables in $\mathbb{D}_H$, Dirichlet independent with the same Dirichlet law. If we denote by $\Sigma$ the covariance operator of $U_1$, then, $V_n = \frac{U_1 + \ldots + U_n}{\sqrt{n}}$ converges in Dirichlet law towards $\hat{S} = (H, \mathcal{B}(H), \nu, \hat{D}, \hat{\Gamma})$ where

i) $\nu$ is a centered Gaussian measure on $H$ with covariance operator $\Sigma$,

ii) $\forall F \in C^1(H, \mathbb{R}) \cap \text{Lip}$, $F \in \hat{D}$ and

$$\hat{\mathbb{E}}[F] = \frac{1}{2} \int_{H^2} < F'(x), y >^2 d\mu(x, y)$$
where \( \mu \) is a centered Gaussian measure on \( H^2 \) with covariance operator \( K \) defined by \( \forall x = (x_1, x_2), y = (y_1, y_2) \) in \( H^2 \)

\[
<Kx, y>_{H^2} = \langle \Sigma x_1, y_1 \rangle + 2\mathcal{E}[< U_1, x_2 >, < U_1, y_2 >],
\]

iii) the form \((C^1(H, \mathbb{R}) \cap \text{Lip}, \mathcal{E})\) is closable. Its smallest closed extension is denoted by \((\mathcal{D}, \mathcal{E})\) and admits a squared field operator \( \hat{\Gamma} \).

Thus, \( V_n \) converges in Dirichlet law towards a structure of the Ornstein-Uhlenbeck type.

**Remark 2.** The operator \( K \) is in the trace class because \( \Sigma \) is a covariance operator and \( U_1 \in \mathcal{D}_H \) (remark 1).

Before proving the theorem, we show that the hypothesis of Dirichlet independence allow us to consider that the variables \( U_i \) are of the form \( U_i \circ g_i \) where the \((g_i)_{i \in \mathbb{N}^*}\) are the coordinate mappings of a product of error structures.

Define \( s = (\Omega, A, P, dl, \gamma) = (\mathcal{W}, m, \mathcal{D}, \Gamma)^{\mathbb{N}^*} \) and \( e[.,.] = \frac{1}{2} \mathcal{E}_P[\gamma[.,.]] \). We set

\[
\ell^2(\mathcal{H}) = \{(h_n)_{n \in \mathbb{N}^*}; \forall i \in \mathbb{N}^*, h_i \in \mathcal{H} \text{ and } \sum_{i=1}^{\infty} \| h_i \|^2 < \infty \}.
\]

Classically, we can build a gradient operator \( \tilde{\nabla} \) for \( s \) setting

\[
\tilde{\nabla} : F \in dl \mapsto (\ldots, \nabla_{[i]}[F], \ldots) \in L^2(P; \ell^2(\mathcal{H}))
\]

where \( \nabla_{[i]} \) means that the operator \( \nabla \) acts on the \( i \)-th variable of \( F \). In the same way, if \((\hat{\Omega}, \hat{A}, \hat{P})\) denotes a copy of \((\Omega, A, P)\), we obtain the following derivative operator \( ' \) for \( s \):

\[
\forall F \in dl, \quad F'(\omega, \hat{\omega}) = \sum_{i \in \mathbb{N}^*} J[\nabla_{[i]}[F](\omega)](\hat{\omega}_i) \in L^2(P \otimes \hat{P})
\]

where the serie converges in \( L^2(\hat{P}) \). For all \( i \in \mathbb{N}^* \), we put \( X_i : \omega = (\omega_j)_{j \in \mathbb{N}^*} \in \Omega \mapsto U_i(\omega_i) \). Then, the following lemma is easily derived.

**Lemma 1.**

a) \( \forall i \in \mathbb{N}^* \), \( X_i \in dl_H \) and \( (X_i)'(\omega, \hat{\omega}) = U_i^\#(\omega_i, \hat{\omega}_i) \).

b) The variables \((X_i)_{i \in \mathbb{N}^*}\) are Dirichlet independent with the same Dirichlet law.

Let us put \( Z_n = \frac{X_1 + \ldots + X_n}{\sqrt{n}} \). Now we state a result that will be used hereafter.

**Lemma 2.** \( \forall F \in C^1(H, \mathbb{R}) \cap \text{Lip}, \) one has \( \mathcal{E}[F(V_n)] = e[F(Z_n)] \).

Proof. Let us first suppose that \( F \) is a cylinder function. For lighten the notation we only consider the case \( F = f(< x, . >) \) where \( x \in H \) and \( f \in C^1(\mathbb{R}, \mathbb{R}) \cap \text{Lip} \).
By the functional calculus one has

\[ e[F(Z_n)] = \frac{1}{2n} \int \sum_{i,j=1}^{n} f^2(<Z_n,x>) \gamma[<x,X_i>,<x,X_j>]dP. \]

For every \( k \), \( X_k \) is Dirichlet independent of \( (X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_n) \). Then, proposition 2 yields

\[ e[F(Z_n)] = \frac{1}{2n} \int \sum_{i=1}^{n} f^2(<Z_n,x>) \mathbb{E}[\gamma[<x,X_i>] | X_i]dP. \]

Since \( (X_1, \ldots, X_n) \) has the same law than \( (U_1, \ldots, U_n) \) and \( \gamma[<x,X_i>](\omega) = \Gamma[<x,U_i>](\omega_i) \), the conclusion holds.

For the general case, let us denote by \( (e_i)_{i \in \mathbb{N}} \) an orthonormal basis of \( H \) and by \( P_n \) the projection on the vector space spans by \( (e_0, \ldots, e_n) \). Since \( Z_n \in dH \) and \( V_n \in D_H \), using the same argument as in the proof of proposition 1 one sees that \( \forall F \in C^1(H,\mathbb{R}) \cap Lip \),

\[ e[F \circ P_k(Z_n)] \rightarrow_{k \rightarrow \infty} e[F(Z_n)] \text{ and } \mathcal{E}[F \circ P_k(V_n)] \rightarrow_{k \rightarrow \infty} \mathcal{E}[F(V_n)]. \]

The result follows by uniqueness of the limit.

Now we can come back to the proof of the theorem.

Proof of theorem 1. The convergence in distribution of the sequence \( (V_n)_m \) towards \( \nu \) is a consequence of the classical central limit theorem for Hilbert valued random variables ([13]).

First, we are going to show that \( \forall F \in C^1(H,\mathbb{R}) \cap Lip \), one has

\[ \mathcal{E}[F(V_n)] \rightarrow_{n \rightarrow \infty} \hat{\mathcal{E}}[F]. \]

In this way, by lemma 2, we study the asymptotic behavior of \( e[F(Z_n)] \). Since \( Z_n \in dH \), one obtains by proposition 1

\[ 2e[F(Z_n)] = \int \int_{\Omega} \langle F'(Z_n),Z_n' \rangle^2 dPd\hat{P}. \]

Since the pairs \( (X_i,X_i') \) are i.i.d, the central limit theorem in Hilbert spaces ensures that \( (Z_n,Z_n') \) converges in distribution in \( H^2 \) towards a centered Gaussian measure denoted by \( \mu \) with a covariance operator \( K \) fulfilling \( \forall x = (x_1,x_2), y = (y_1,y_2) \) in \( H^2 \)

\[ <Kx, y>_{H^2} = <\Sigma x_1, y_1> + 2\mathcal{E}[<U_1, x_2>, <U_1, y_2>]. \]
Moreover, using the independence of the \((X_i, X_i')\), it follows that
\[
\mathbb{E}_P[\|Z_n\|^2 + \|Z_n'\|^2] = \mathbb{E}_m[\|U_1\|^2] + \mathbb{E}_m[\|U_1'\|^2] = \int_{H^2} \|x\|^2_{H^2} d\mu(x),
\]
thus, the \(\|Z_n, Z_n'\|_{H^2}^2\)'s are uniformly integrable.

Since the function \(\phi : (x, y) \in H^2 \mapsto \langle F'(x), y \rangle^2\) is continuous and satisfies
\[
\phi(x, y) \leq c \| (x, y) \|^2_{H^2},
\]
where \(c\) is a constant, one obtains
\[
2\mathbb{E}[F(V_n)] \rightarrow \int_{H^2} < F'(x), y >^2 d\mu(x, y).
\]

To conclude, it remains to show the closability of the form \(\hat{\mathcal{E}}\) defined on \(C^1(H, \mathbb{R}) \cap \text{Lip}\) by (4). The proof is based on three lemmas which proves are left in appendix.

**Lemma 3.** Let \(\beta\) be an orthonormal basis of \(H\), one has
\[
(C^1(H, \mathbb{R}) \cap \text{Lip}, \hat{\mathcal{E}}) \text{ closable} \iff (A_\beta, \hat{\mathcal{E}}) \text{ closable}
\]
where \(A_\beta = \bigcup_{p \in \mathbb{N}^*} \{f(<e_1, \ldots, e_p, \ldots>); (e_1, \ldots, e_p) \in \beta^p, f \in C^1(\mathbb{R}^p, \mathbb{R}) \cap \text{Lip}\}\). Moreover, when one of the assertions is fulfilled their smallest closed extensions coincide.

Since \(\Sigma\) is a covariance operator, it is positive and belongs to the trace class. Thus ([12]), there exists an orthonormal basis \(\beta_0 = (e_i)_{i \in \mathbb{N}}\) of \(H\) constituted of eigenvectors of \(\Sigma\). According to the preceding lemma, we only have to prove that \((A_{\beta_0}, \hat{\mathcal{E}})\) is closable. Let us denote by \((\sigma_i^2)_{i \in \mathbb{N}}\) the corresponding eigenvalues, hence, the sequence \((<e_1, \ldots, >)_{i \in \mathbb{N}}\), defined on \((H, B(H), \nu)\), is a sequence of independent and centered Gaussian variables on \(\mathbb{R}\) with variances \((\sigma_i^2)_{i \in \mathbb{N}}\).

From \(U_1 \in D_H\), it follows from proposition 1 that the bilinear operator
\[
T : \left( H^2 \rightarrow \mathbb{R} \right.
\]
\[
(x, y) \mapsto \mathcal{E}[< x, U_1 >, < y, U_1 >]
\]
is continuous because
\[
\mathcal{E}[< x, U_1 >, < y, U_1 >] \leq \| x \| \| y \| \mathbb{E}_m[\| U_1' \|^2].
\]
Therefore, there exists a bounded operator \(C : H \rightarrow H\) such that \(T(x, y) = < Cx, y >\). The operator \(C\) is clearly selfadjoint and positive. Let us define
\[
D_C = C^{1/2} D : A_{\beta_0} \rightarrow L^2(\nu; H)
\]
where \(D\) stands for the Fréchet derivative in \(H\). Thus, we have the following equality: \(\forall F \in A_{\beta_0}, \)
\[
\hat{\mathcal{E}}[F] = \int_{H} \| D_C[F] \|^2 d\nu.
\]
Therefore, the closability of the form \((\mathcal{A}_{\beta_0}, \mathcal{E})\) is equivalent to the closability of \(D_C\) in \(L^2(\nu; H)\). We have formulated our closability problem in terms of directional gradient in the sense of Goldys and al in [10]. According to the following lemma we can impose that \(\forall i \in \mathbb{N}, \sigma_i^2 > 0\).

**Lemma 4.** When we study the closability of \((\mathcal{A}_{\beta_0}, \mathcal{E})\) we can suppose that \(\Sigma\) is injective.

Thus, the operator \(V = C^\frac{1}{2}\Sigma^{-\frac{1}{2}}\) is well defined on \(\text{dom}(V) = \Sigma^\frac{1}{2}(H)\) and we have the following result:

**Lemma 5.** \((\mathcal{A}_{\beta_0}, D_C)\) closable \(\iff (\text{dom}(V), V)\) closable.

We show the closability of \(V\): Let \((x_n)_{n \in \mathbb{N}}\) be a sequence in \(\Sigma^\frac{1}{2}(H)\) fulfilling \(x_n \rightarrow 0\) and \(V x_n \rightarrow u\). For all \(n\), we denote by \(h_n\) the unique element in \(H\) such that \(x_n = \Sigma^\frac{1}{2}(h_n)\). Then, it follows that \(\int_W < h_n, U_1 >^2 dm \rightarrow 0\) and \(\mathcal{E}[< h_n - h_m, U_1 >] \rightarrow 0\). Since \(\hat{S}\) is an error structure, the closedness property implies \(\mathcal{E}(< h_n, U_1 >) = \| V x_n \|^2 \rightarrow 0\) and \(u = 0\). Thus, the theorem is proved.

4. Concluding remarks

We use the notations of the preceding proof. Let us consider \(\Lambda\) the set of subsets of \(\mathbb{N}\). If \(u = \{i_1, \ldots, i_n\} \in \Lambda\), \(\pi_u\) denotes the canonical projection from \(H\) into \(\text{vect}(e_{i_1}, \ldots, e_{i_n})\) and \(\phi_u\) the natural homeomorphism between \(\mathbb{R}^n\) and \(\text{vect}(e_{i_1}, \ldots, e_{i_n})\). Let us define the following error structure

\[
\hat{S}_u = (\mathbb{R}^n, B(\mathbb{R}^n), \mathcal{N}(0, \sigma_{i_1}) \otimes \ldots \otimes \mathcal{N}(0, \sigma_{i_n}), \hat{D}_u, \hat{\Gamma}_u)
\]

with \(\forall f \in C^1(\mathbb{R}^n, \mathbb{R}) \cap \text{Lip},\)

\[
\hat{\Gamma}_u[f] = 2 \sum_{k,l=1}^{n} \frac{\partial f}{\partial x_k} \frac{\partial f}{\partial x_l} \mathcal{E}[< e_{i_k}, U_1 >, < e_{i_l}, U_1 >]
\]

and where \(\hat{D}_u\) is the domain of the smallest closed extension of \((C^1(\mathbb{R}^n, \mathbb{R}) \cap \text{Lip}, \hat{\mathcal{E}}_u))\).

We can see that \((\phi_u)_*\hat{S}_u)_{u \in \Lambda}\) is a projective system of error structures in the sense of [1], p.206. Since \((\pi_u)_*\hat{S} = (\phi_u)_*\hat{S}_u\), this projective system has a limit which is none other than \(\hat{S}\). Thus, our result may be seen as the projective limit of the result of Bouleau and Hirsch in finite dimension ([1], p.220). Moreover, by the so-called Kwapien theorem ([13]), it is easy to see that theorem 1 extends to Banach spaces having type 2 and cotype 2. The
natural question arises of the extension of such a result in more general settings. Unfortunately, the classical conditions for the central limit theorem to hold in Banach spaces having finite type and cotype ([13]) or in the Wiener space ([11]) seem to be, for the moment, insufficient to overcome the lack of orthogonality that is the keystone of our proof.

5. Appendix: Lemmata

5.1. Proof of lemma 3.

The implication $i \Rightarrow ii$ is obvious. For the converse suppose that $(A_{\beta}, \hat{E})$ is closable. We denote by $\hat{D}_{A_{\beta}}$ the domain of its smallest closed extension. We can see that $(\hat{D}_{A_{\beta}}, \hat{E})$ possesses a gradient with values in $L^2(\nu; H)$ (which is the smallest closed extension of $(D_{C}, A_{\beta})$). Let us denote by $J$ the canonical isometry between $H$ and a copy $\hat{H}$ of $H$ and by $(\hat{D}_{A_{\beta}})_H$ the associated vectorial domain. According to remark 1, the identity mapping of $H$ belongs to $(\hat{D}_{A_{\beta}})_H$. From proposition 1, $C^1(H, \mathbb{R}) \cap Lip \subset \hat{D}_{A_{\beta}}$ thus $(C^1(H, \mathbb{R}) \cap Lip, \hat{E})$ is closable and $\hat{D} \subset \hat{D}_{A_{\beta}}$. The result follows.

5.2. Proof of lemma 4.

We set $N = \{e_i \in \beta_0 | \Sigma(e_i) = 0\}$. Suppose that $e_{i_1} \in N$. Using the locality of the form $E$ ([1], p.28) one obtains $E[<e_{i_1}, U_1>] = 0$. Thus, if $F = f(<e_{i_1}, .. , <e_{i_p}, .. >) \in A_{\beta_0}$ one has

$$\hat{E}[F] = \sum_{k,l=2}^p \int_H f_k^* f_l^*(0, <e_{i_2}, .. , <e_{i_p}, .. >) E[<e_{i_k}, U_1>, <e_{i_l}, U_1>] \, d\nu.$$  

Putting

$A_{\beta_0 \setminus N} = \{F(<e_{i_1}, .. , <e_{i_n}, .. >) ; \, n \in \mathbb{N}, \, (e_{i_1}, .. , e_{i_n}) \in \beta_0 \setminus N, \, F \in C^1 \cap Lip\}$

we deduce from (6) that

$(A_{\beta_0 \setminus N}, \hat{E})$ closable $\Leftrightarrow (A_{\beta_0}, \hat{E})$ closable.

Since the restriction of $\Sigma$ to vect$(\beta_0 \setminus N)$ is injective, this yields the conclusion.

5.3. Proof of lemma 5.

The main ideas of the proof are taken from [10]. Let us define

$\mathcal{P}(V^*) = \{Fk | F \in A_{\beta_0}, k \in \text{dom}(V^*)\} \subset L^2(\nu; H).$

In a natural way, we can extend $D_C$ to $\mathcal{P}(V^*)$ putting

$$D_C[Fk] = D_C[F] \otimes k$$
where $\otimes$ is the tensor product on $H$. Using [10], p.4, we show that the operator

$W[\psi](x) = -\text{trace}(D_C[\psi](x)) + <\Sigma^{-\frac{1}{2}}x, V^*\psi(x)>$, $\text{dom}(W) = \mathcal{P}(V^*)$

is well defined and that $(W, \text{dom}(W))$ and $(D_C, A_{\beta_0})$ are adjoint to each other.

If we assume that $V$ is closable, a classical result ([11], theo. 5.28) gives that $\text{dom}(V^*)$ is dense in $H$. Thus, $\text{dom}(W)$ is dense in $L^2(\nu; H)$. Since $W$ and $D_C$ are adjoint, $D_C$ is closable.

For the reciprocal, let $(h_n)_{n \in \mathbb{N}}$ be a sequence in $H$ such that

$h_n \rightarrow 0$, $Vh_n \rightarrow u$.

Let $f$ be in $C^1(\mathbb{R}, \mathbb{R}) \cap \text{Lip}$ (with a Lipschitz constant equals to $K$) with $f(0) = 0$ and $f'(0) \neq 0$. One has

$$\int |f(<x, \Sigma^{-\frac{1}{2}}h_n>)|^2d\nu(x) \leq K \|h_n\|^2,$$

hence, $f(<., \Sigma^{-\frac{1}{2}}h_n>) \rightarrow 0$ in $L^2(\nu)$. Moreover

$D_C[f(<., \Sigma^{-\frac{1}{2}}h_n>)](x) - f'(0)u = [f'(\langle x, \Sigma^{-\frac{1}{2}}h_n \rangle) - f'(0)]Vh_n + [Vh_n - u]f'(0)$.

Since $\|Vh_n\|$ is bounded,

$D_C[f(<., \Sigma^{-\frac{1}{2}}h_n>)] \rightarrow f'(0)u$ in $L^2(\nu; H)$.

From lemma 3, $(D_C, A_{\beta_0})$ closable $\Rightarrow$ $(D_C, C^1(H, \mathbb{R}) \cap \text{Lip})$ closable. Thus, $u = 0$. 
References


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