DONSKER THEOREM
and
DIRICHLET FORMS

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Study plan

I) Intuitive approach and extension tool

1) Propagation of small errors : Intuitive notion of error structure

2) An extension tool : The language of Dirichlet forms

3) Image and infinite products of error structures

4) Examples

5) Identification of error structures

II) Vectorial domain of a Dirichlet form (Feyel-La Pradelle)

III) Extensions of the Donsker invariance principle

I_1 : Intuitive notion of error structures

Let us consider a quantity C able to be measured by an experimental device which result exhibits an error denoted by $\triangle C$.

Classically we will consider that those quantities are represented by <u>random variables</u> (generally correlated).

• Probabilistic approach :

It is required to know the law of the pair $(C, \triangle C)$ to model the experiment.

Error propagation $\Leftrightarrow$ Computation of images of probability measures

• Intermediate approach :

H$_1$: We suppose that $var[\triangle C \mid C]$ is known and that $\mathbb{E}[\triangle C \mid C] = 0$.

H$_2$: **<u>Small errors</u> :** $\triangle C = \varepsilon Y$ (Y bounded, ε size parameter).

The study of error propagation becomes easier.

Let $(f_i)_{i\in\mathbb{N}^*}$ of classe C^3 with bounded derivatives. We set

$$\mathbf{b_0 = 0} \quad \sigma_0^2 = \mathbf{var}[\triangle \mathbf{C} \mid \mathbf{C}]$$

and $\forall n \in \mathbb{N}^*$

$$\mathbf{b_n} = \mathbb{E}[\triangle \mathbf{g_n(C)} \mid \mathbf{C}] \quad \text{and} \quad \sigma_{\mathbf{n}}^2 = \mathbf{var}[\triangle \mathbf{g_n(C)} \mid \mathbf{C}]$$

where $g_n = f_n o \dots o f_1$. Putting $x_n = g_n(C)$

$$\sigma_{\mathbf{n}}^2 = (\mathbf{f'_n})^2[\mathbf{x_{n-1}}] \ \sigma_{\mathbf{n-1}}^2 + \varepsilon^3 \ \mathbf{0(1)}$$

$$\mathbf{b_n} = \mathbf{f'_n}[\mathbf{x_{n-1}}] \ \mathbf{b_{n-1}} + \frac{\mathbf{1}}{\mathbf{2}} \ \mathbf{f''_n}[\mathbf{x_{n-1}}] \ \sigma_{\mathbf{n-1}}^2 + \varepsilon^3 \ \mathbf{0(1)}.$$

We can see that the calculus on the variances is a first order calculus and does not involve the biases when the calculus on the biases is of the second order and involves the variances.

• Intuitive definition

On the space $(\mathbb{R}, Bor(\mathbb{R}), law\ of\ C)$, we introduce the operator Γ^C, called the quadratic error operator, defined by

$$\Gamma^C[f](x) = \underset{\varepsilon \to 0}{Lim} \frac{var[\triangle f(C) \mid C = x]}{\varepsilon^2}.$$

For $f = Id$,

$$\Gamma^C[Id](x) = \underset{\varepsilon \to 0}{Lim} \frac{var[\triangle C \mid C = x]}{\varepsilon^2} = var[Y|C = x].$$

Γ^C naturally polarizes into a bilinear, symmetric, positive operator fulfilling the following functional calculus : If $F \in C^2(\mathbb{R}^2)$ with bounded partial derivatives

$$\Gamma^C[F(f,g)] = (F_1')^2(f,g)\Gamma^C[f] + (F_2')^2(f,g)\Gamma^C[g] + 2F_1'(f,g)F_2'(f,g)\Gamma^C[f,g].$$

Propagation "à la Gauss" (1821)

The term $(\mathbb{R}, \mathrm{Bor}(\mathbb{R}), \mathrm{law\ of\ C}, \Gamma^{\mathrm{C}})$ is called an error structure.

I_2 : The language of Dirichlet forms

From now on, an error structure is a term $(W, \mathcal{W}, m, \mathbb{D}, \Gamma)$ where $(W, \mathcal{W}, m)$ is a probability space, $\mathbb{D}$ is a dense vector subspace of $L^2(m)$ and Γ is a positive symmetric bilinear map from $\mathbb{D} \times \mathbb{D}$ into $L^1(m)$ fulfilling :

1) **the functional calculus** of class $C^1 \cap Lip$ i.e. if $U = (U_1, \ldots, U_n) \in \mathbb{D}^n$, $F \in C^1(\mathbb{R}^n) \cap Lip$ then $F(U_1, \ldots, U_n) \in \mathbb{D}$ and

$$\Gamma[F(U_1, \ldots, U_n)] = \sum_{i,j} F'_i(U) F'_j(U) \Gamma[U_i, U_j]$$

where $\Gamma[V] := \Gamma[V, V]$ (V in $\mathbb{D}$).

2) $1 \in \mathbb{D}$, (this implies $\Gamma[1] = 0$).

3) the bilinear form defined on $\mathbb{D} \times \mathbb{D}$ by $\mathcal{E}[F, G] = \frac{1}{2} \int \Gamma[F, G] dm$ is closed i.e. $\mathbb{D}$ is complete under the norm

$$\| \,.\, \|_{\mathcal{E}} = (\| \,.\, \|^2_{L^2(m)} + \mathcal{E}[.])^{\frac{1}{2}}.$$

COMMENTS

• **Natural extension** of the intuitive notion (property 1)

• **Easy to handle** (Image, products)

• Property (3) ensures that the domain $\mathbb{D}$ is preserved by Lipschitz functions : if F is a contraction then for $U \in \mathbb{D}$ one has $F(U) \in \mathbb{D}$ and

$$\Gamma[F(U)] \leq \Gamma[U].$$

• **Natural calculus for biases** : To the Dirichlet form $\mathcal{E}$ corresponds a self adjoint operator A with a domain $D(A) \subset \mathbb{D}$ which statisfies

$$A[F(U)] = F'(U)A[U] + \frac{1}{2}F''(U)\Gamma[U]$$

for $U \in D(A)$, $\Gamma[U] \in L^2(m)$ and $F : \mathbb{R} \to \mathbb{R}$ of classe C^2 with bounded derivatives.

I_3 : Image and infinite product

Remark : *If U is a random variable on $(W, \mathcal{W}, m)$ the image probability $U_* m$ is defined by*

$$\mathbb{E}_{U_* m}[F] = \mathbb{E}_m[F(U)].$$

• Image structure

Let $S = (W, \mathcal{W}, m, \mathbb{D}, \Gamma)$ be an error structure and $U \in \mathbb{D}^d$. The bilinear form

$$(C^1(\mathbb{R}^d) \cap Lip, (F, G) \mapsto \mathcal{E}[F(U), G(U)])$$

is closable.

Its smallest closed extension $(\mathbb{D}_U, \mathcal{E}_U)$ is a local Dirichlet form possessing a squared field operator Γ_U.

The error structure

$$\mathbf{U_* S} = (\mathbb{R}^\mathbf{d}, \mathcal{B}(\mathbb{R}^\mathbf{d}), \mathbf{U_* m}, \mathbb{D}_\mathbf{U}, \mathbf{\Gamma_U})$$

is called **the image of S by U** or the $\mathbb{D}$-**law of U**.

• Infinite product

Let $S_n = (W_n, \mathcal{W}_n, m_n, \mathbb{D}_n, \Gamma_n)$, $n \geq 0$, be a family of error structures. The product structure

$$(W, \mathcal{W}, m, \mathbb{D}, \Gamma) = \prod_{n=0}^{\infty} S_n$$

is defined by $(W, \mathcal{W}, m) = (\prod_{n=0}^{\infty} W_n, \prod_{n=0}^{\infty} \mathcal{W}_n, \otimes_{n=0}^{\infty} m_n)$ with an explicit domain $\mathbb{D}$ and $\forall F \in \mathbb{D}$,

$$\Gamma[F] = \sum_{n=0}^{\infty} \Gamma_n[F].$$

Application : The random variables $U_1 \in \mathbb{D}^p$ and $U_2 \in \mathbb{D}^m$ are said to be $\mathbb{D}$-independent if

$$(U_1)_* S \otimes (U_2)_* S = (U_1, U_2)_* S.$$

It is naturally the case of the coordinate mappings of a product of error structures.

I_4 : Examples of error structures

- **On $\mathbb{R}$ they are well known (Hamza's theorem)**

Ornstein-Uhlenbeck structure on $\mathbb{R}$:

$$(\mathbb{R}, \mathcal{B}(\mathbb{R}), m = \mathcal{N}(0,1), H^1(m), u \mapsto (u')^2).$$

- **On the Wiener space**

The Ornstein-Uhlenbeck structure on the Wiener space $\mathcal{C} = (C_0([0,1], \mathbb{R}), \|\|_\infty)$ equipped with the Wiener measure μ :

$$S_{OU} = (\mathcal{C}, \mathcal{B}(\mathcal{C}), \mu, \mathbb{D}_{OU}, \Gamma_{OU})$$

where $\forall\ F = f(\lambda_1, .., \lambda_n)$ regular cylindrical function,

$$\Gamma_{OU}[F] = \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}(\lambda_1, \ldots, \lambda_n) \ < \lambda_i, \lambda_j >_{L^2([0,1],dx)} .$$

I_5 Identification of error structures

How to obtain an error structure from experiment ?

• **In finite dimension :** Error structures are linked to parametric estimation via the notion of **Fisher information**.

Cf : *N.Bouleau et Ch.Chorro, "Error structure and parameter estimation", note C.R.A.S, Février 2004.*

• **In infinite dimension** (especially for the Wiener space), the preceding approach is meaningless. In this case we study the extension to error structures of the main limit theorems of probability theory :

- -Central limit theorem in Hilbert spaces
- -**Donsker invariance principle**
- -Approximation of the Brownian motion by means of random series

For the second point a question naturally arises :

Can we propose an efficient extension of the notion of the domain of a Dirichlet form for random variables with values in an **infinite dimensional** space ?

II Vectorial domain of a Dirichlet form

A] Hypothesis (G), derivative operator

a) Hypothesis (G)

We will say that $S = (W, \mathcal{W}, m, \mathbb{D}, \Gamma)$ fulfills property (G) if there exist a separable Hilbert space $(\mathcal{H}, \|\|_{\mathcal{H}})$ and an operator ∇ from $\mathbb{D}$ into $L^2(m; \mathcal{H})$ such that

$$\forall X \in \mathbb{D} \qquad \| \nabla X \|_{\mathcal{H}}^2 = \Gamma[X].$$

Remark : (G) is fulfilled whenever $\mathbb{D}$ is separable (Mokobovski).

b) The derivative operator

let $(\tilde{W}, \widetilde{\mathcal{W}}, \tilde{m})$ be a copy of $(W, \mathcal{W}, m)$ and J an isometry from $\mathcal{H}$ into $L^2((\tilde{W}, \widetilde{\mathcal{W}}, \tilde{m}))$ such that $\forall h \in \mathcal{H}$, $\mathbb{E}_{\tilde{m}}[J(h)] = 0$. The derivative operator # is defined in the following way, $\forall U \in \mathbb{D}$,

$$U^{\#} = J(\nabla U) \in L^2(m \otimes \tilde{m}).$$

c) Example

The Ornstein-Uhlenbeck error structure on the Wiener space S_{OU} fulfills property (G) with $\mathcal{H} = L^2([0,1])$ and

$$\nabla_{\mathrm{OU}}\left[\int_0^1 \mathbf{h(s)dB_s}\right] = \mathbf{h}, \quad \forall\ \mathbf{h} \in \mathcal{H}.$$

If we denote by $(\tilde{\mathcal{C}}, \widetilde{\mathcal{B}(\mathcal{C})}, \tilde{\mu})$ a copy of $(\mathcal{C}, \mathcal{B}(\mathcal{C}), \mu)$ and $(\tilde{B}_t)_{t\in[0,1]}$ the associated Brownian motion, a derivative operator is given by

$$\left(\int_0^1 \mathbf{h(s)dB_s}\right)^{\#_{\mathrm{OU}}} = \int_0^1 \mathbf{h(s)d\tilde{B}_s}.$$

Using the Ito calculus it is easy to compute the derivative of the solutions of regular SDE. It allows to study the sensitivity of financial models to the variations of functional parameters.

Nicolas Bouleau, Error Calculus and Path Sensitivity in Financial Models, Mathematical finance 13, 115-134, 2003.

B] Vectorial domain

Let B be a Banach space having a Schauder basis. Let $S = (W, \mathcal{W}, m, \mathbb{D}, \Gamma)$ be an error structure that owns a gradient.

Aim : Extend the notion of domain to B-valued random variables.

Definition :(Feyel-La Pradelle) Let us denote by $\mathbb{D}_B$ the vector space of random variables U in $L^2(m; B)$ such that there exists g in $L^2(m \otimes \tilde{m}; B)$ so that

$$\forall \lambda \in B',\ < \lambda, U > \in \mathbb{D} \quad and \quad < \lambda, U >^{\#} = < \lambda, g > .$$

We then put $g = U^{\#}$ and one equips $\mathbb{D}_B$ with the norm

$$\| U \|_{\mathbb{D}_B} = \left(\| U \|^2_{L^2(m;B)} + \frac{1}{2} \| U^{\#} \|^2_{L^2(m \times \tilde{m};B)} \right)^{\frac{1}{2}} .$$

(Thus $\mathbb{D}_{\mathbb{R}} = \mathbb{D}$).

Some properties of the domain naturally extend to the vectorial domain.

1) **Functional calculus**

Proposition : Let F be a Lipschitz function from B into $\mathbb{R}$.

a) If $U \in \mathbb{D}_B$,

$$F(U) \in \mathbb{D} \quad \text{and} \quad \Gamma[F(U)] \leq K^2 \mathbb{E}_{\tilde{m}}[\| U^{\#} \|_B^2].$$

b) Moreover if F is of class C^1,

$$F(U)^{\#} = < F'(U), U^{\#} > . \quad (*)$$

Remark : when $B = \mathbb{R}^p$

$$(*) \Leftrightarrow F(U)^{\#} = \sum_{i=1}^{p} F_i'(U) U_i^{\#}$$

2) **Image of an error structure by an element of the vectorial domain**

Let $S = (W, \mathcal{W}, m, \mathbb{D}, \Gamma)$ be an error structure and $U \in \mathbb{D}_B$. The bilinear form

$$(C^1(B, \mathbb{R}) \cap Lip, (F, G) \mapsto \mathcal{E}[F(U), G(U)])$$

is closable.

Its smallest closed extension $(\mathbb{D}_U, \mathcal{E}_U)$ is a local Dirichlet form owning a squared field operator Γ_U.

The structure

$$U_*S = (B, \mathcal{B}(B), U_*m, \mathbb{D}_U, \Gamma_U)$$

is called **the image of S by U or the $\mathbb{D}$-law of U**.

Example :

The solutions of regular SDE are in $(\mathbb{D}_{\mathrm{OU}})_{\mathcal{C}}$.

In particular for $h \in L^2([0,1])$,

$$\int_0^{\cdot} h(s)dB_s \in (\mathbb{D}_{OU})_{\mathcal{C}} \quad and \quad \left(\int_0^{\cdot} h(s)dB_s\right)^{\#} = \int_0^{\cdot} h(s)d\tilde{B}_s.$$

We denote by

$$S_{OU}^h = (\int_0^{\cdot} h(s)dB_s)_* S_{OU}$$

the image structure. Its associated Dirichlet form $\mathcal{E}_{OU}^h$ satisfies $\forall F \in C^1(\mathcal{C}, \mathbb{R}) \cap Lip$

$$\mathcal{E}_{OU}^h[F] = \int_{\mathcal{C}} \int_{\mathcal{C}} \left\langle F'\left(\int_0^{\cdot} h(s)dB_s\right), \int_0^{\cdot} h(s)d\tilde{B}_s\right\rangle^2 d\mu d\tilde{\mu}.$$

C] Convergence in $\mathbb{D}$-law

Definition : We say that a sequence $(U_n)_{n\in\mathbb{N}} \in \mathbb{D}_B$ converges in $\mathbb{D}$-law if there exists an error structure $\tilde{S} = (B, \mathcal{B}(B), P, \tilde{\mathbb{D}}, \tilde{\Gamma})$ such that :

i) $(U_n)_* m \to P$ weakly in B,

ii) $C^1(B,\mathbb{R}) \cap Lip \subset \tilde{\mathbb{D}}$ and $\forall F \in C^1(B,\mathbb{R}) \cap Lip$, $\mathcal{E}[F(U_n)] \to \tilde{\mathcal{E}}[F]$.

• **To prove the convergence in Dirichlet law of a sequence $(U_n)_{n\in\mathbb{N}} \in \mathbb{D}_B$ we have to show three points :**

- $-$Study the convergence of (U_n) towards P
- $-\forall F \in C^1(B,\mathbb{R}) \cap Lip,$ study the convergence of $\mathcal{E}[F(U_n)]$ towards a limit denoted by $\tilde{\mathcal{E}}[F]$
- $-$Study the closability of the obtained error pre-structure

• **Of course this notion is stable by functions of class C^1 and Lipschitz.**

Examples :

- If $\|U_n - U\|_{\mathbb{D}_B} \underset{n\to\infty}{\longrightarrow} 0$, (U_n) converges in $\mathbb{D}$-law towards U_*S.

- Let H be a separable Hilbert space.

Let (U_n) be a sequence of random variables in $\mathbb{D}_H$, centered, $\mathbb{D}$-independent with the same $\mathbb{D}$-law. Then, $\frac{1}{\sqrt{n}}\sum_{k=1}^{n} U_k$ converges in Dirichlet law towards an error structure of the Ornstein-Uhlenbeck type.

C. Chorro : On an extension of the central limit theorem to Dirichlet forms, *Cahiers de la mse, 2004.80, Univ. Paris 1, 2004.*

III Extensions of Donsker theorem

Reminder : Let $(U_k)_{k\in\mathbb{N}^*}$ be a sequence of i.i.d random variables defined on $(W, \mathcal{W}, P)$, centered and reduced. We set $\forall t \in [0,1]$

$$X_n(t) = \frac{1}{\sqrt{n}} \left(\sum_{k=1}^{[nt]} U_k + (nt - [nt]) U_{[nt]+1} \right)$$

The Donsker invariance principle ensures the weak convergence in $\mathcal{C}$ of (X_n) towards the Wiener measure μ.

Here we suppose that the U_k's are erroneous (errors being modelized by error structures) with a condition of independence and stationarity for the errors.

In other words : **the $\mathrm{U_k}$'s are the coordinate maps of the following product**

$$S = (W, \mathcal{W}, P, \mathbb{D}, \Gamma) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda, d, \gamma)^{\mathbb{N}^*} = s^{\mathbb{N}^*}.$$

The error structure $s = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda, d, \gamma)$ being such that the identity $i \in d$, $\mathbb{E}_\lambda[i] = 0$, $\mathbb{E}_\lambda[i^2] = 1$ and $e[i] = 1$.

Thus the U_k's are $i.i.d$ random variables on W with distribution λ fulfilling $U_k \in \mathbb{D}$ and

$$\Gamma[U_n, U_m] = \delta_n^m \, \gamma[i](U_n)$$

The U_k's are $\mathbb{D}$-independent with the same $\mathbb{D}$-law.

HYP : The structure s **fulfills property (G)** and we denote by $'$ a derivative operator built with a copy $(\widetilde{\mathbb{R}}, \widetilde{\mathcal{B}(\mathbb{R})}, \widetilde{\lambda})$ of $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$.

We can deduce classically a derivative operator # for S putting $\forall p \in \mathbb{N}^*$ and $\forall F \in C^1(\mathbb{R}^p, \mathbb{R}) \cap Lip$,

$$\mathbf{F(U_1, \ldots, U_p)^{\#} = \sum_{k=1}^{p} F'_k(U_1, \ldots, U_p)\ i'(U_k, \widetilde{U_k})},$$

the $(\widetilde{U_k})$ being the coordinate mappings of $(\widetilde{W}, \widetilde{\mathcal{W}}, \widetilde{P}) = (\widetilde{\mathbb{R}}, \widetilde{\mathcal{B}(\mathbb{R})}, \widetilde{\lambda})^{\mathbb{N}^*}$

We have the following natural result (*N. Bouleau : Théorème de Donsker et Formes de Dirichlet, to be published in Bull.Scien.Math., 2005*)

> **Proposition : The sequence (X_n) converges in Dirichlet law towards S_{OU}**

Proof : Purely probabilistic extension of the Donsker theorem.

We are going to extend this result to some families of stochastic integrals :

$$\begin{cases} \textbf{A]} \quad \int_0^{\cdot} h(s)dX_n(s), \ h \in L^2([0,1]) \\ \textbf{B]} \quad \int_0^{\cdot} K(s,.)dX_n(s), \ K \textbf{ regular} \\ \textbf{C]} \quad \int_{[0,.]^p} h(x_1,\ldots,x_p)dX_n(x_1)\ldots dX_n(x_p), \ h \textbf{ given by a multimeasure} \end{cases}$$

A] Convergence in Dirichlet law of $\int_0^{\cdot} h(s) dX_n(s)$

Let $h \in L^2([0,1])$, we put

$$Y_n^h(t) = \int_0^t h(s) dX_n(s) = \sqrt{n} \sum_{k=1}^{n} U_k \int_{\frac{k-1}{n}}^{\frac{k}{n}} h(s) I_{[0,t]}(s) ds.$$

Proposition : The sequence (Y_n^h) converges in Dirichlet law towards $S_{OU}^h = (\int_0^{\cdot} h(s) dB_s)_* S_{OU}$

Proof :

a) Weak convergence in $\mathcal{C}$

X_n being a continuous semi-martingale two conditions may be found in the literature depending on the regularity of h

• If h is càd-làg, **condition U.T** of Jakubowski, Mémin and Pagès.

• For general h, convergence of X_n towards B. for the distance in variation.

Although very simple our problem leaves this framework $\Rightarrow$ Direct proof

We first suppose that h is continuous and will conclude by approximations in $L^2([0,1])$.

Let us define

$$\widetilde{Y_n^h}(t) = \frac{1}{\sqrt{n}} \left(\sum_{k=1}^{[nt]} h\left(\frac{k}{n}\right) U_k + (nt - [nt]) h\left(\frac{[nt]+1}{n}\right) U_{[nt]+1} \right).$$

One has $\forall \varepsilon > 0$,

$$P(\|\widetilde{Y_n^h} - Y_n^h\|_\infty > \varepsilon) \underset{n\to\infty}{\to} 0.$$

The weak convergence of $\widetilde{Y_n^h}$ towards $Y^h = \int_0^\cdot h(s)dB_s$ is a consequence of the functional invariance principle of **Lindeberg-Feller** (*D. Dacunha-Castelle, M. Duflo*).

Remark : For $h = I_{\mathbb{R}\setminus\mathbb{Q}}$ one has $\widetilde{Y_n^h} = 0$ and $Y_n^h = X_n$.

For $h \in L^2([0,1])$, let g be a continuous function such that $\|h-g\|_{L^2([0,1])} \leq \varepsilon$.
If $\Phi : \mathcal{C} \to \mathbb{R}$ is a bounded Lipschitz function

$$\mathbb{E}[\Phi(Y_n^h) - \Phi(Y^h)] = \underbrace{\mathbb{E}[\Phi(Y_n^h) - \Phi(Y_n^g)]}_{(1)} + \underbrace{\mathbb{E}[\Phi(Y_n^g) - \Phi(Y^g)]}_{(2)} + \underbrace{\mathbb{E}[\Phi(Y^g) - \Phi(Y^h)]}_{(3)}.$$

(3) : **Doob inequality**

(1) : **Piecewise monotony of** $Y_n^{(h-g)_+}$ **and** $Y_n^{(h-g)_-}$ $\bigoplus$ **Doob inequality**.

(2) : **Preceding result.**

Remark : This result remains valid if we suppose that the (U_k) take values in $\mathbb{R}^p$.

b) Study of the sequence $\mathcal{E}[F(Y_n^h)]$

The process Y_n^h belongs to $\mathbb{D}_{\mathcal{C}}$ with

$$(Y_n^h)^{\#}(t) = \sqrt{n} \sum_{k=1}^{n} U_k^{\#} \int_{\frac{k-1}{n}}^{\frac{k}{n}} h(s) I_{[0,t]}(s) ds.$$

Thus, the functional calculus gives for all $F \in C^1(\mathcal{C}, \mathbb{R}) \cap Lip$,

$$\mathcal{E}[F(Y_n^h)] = \int_W \int_{\widetilde{W}} < F'(Y_n^h), (Y_n^h)^{\#} >^2 d\widetilde{P} dP = \int_W \int_{\widetilde{W}} \phi\left(Y_n^h, (Y_n^h)^{\#}\right) d\widetilde{P} dP$$

where ϕ is **continuous and sub-quadratic.**

The pairs $(U_k, (U_k)^{\#})$ being $i.i.d$, centered with a covariance matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ one obtains

$$(Y_n^h, (Y_n^h)^{\#}) \underset{\mathcal{L}}{\rightarrow} (\int_0^{\cdot} h(s)dB_s, \int_0^{\cdot} h(s)d\widetilde{B}_s).$$

<u>If</u> this weak convergence extends to continuous sub-quadratic functions one has

$$\mathcal{E}[F(Y_n^h)] \underset{n\to\infty}{\rightarrow} \int_{\mathcal{C}}\int_{\mathcal{C}} \left\langle F'\left(\int_0^{\cdot} h(s)dB_s\right), \int_0^{\cdot} h(s)d\tilde{B}_s\right\rangle^2 d\mu d\tilde{\mu} = \mathcal{E}_{OU}^h[F]$$

where $\mathcal{E}_{OU}^h$ is the Dirichlet form associated to $S_{OU}^h = (\int_0^{\cdot} h(s)dB_s)_* S_{OU}$.

Proposition : Let $\phi : \mathcal{C} \to \mathbb{R}$ be a continuous function such that $\forall x \in \mathcal{C}$, $|\phi(x)| \leq K(1 + \|x\|_\infty^2)$ one has

$$\mathbb{E}_P[\phi(Y_n^h)] \underset{n\to\infty}{\to} \mathbb{E}_\mu[\phi(Y^h)].$$

Proof : We only have to show the **uniform integrability** of the $\|Y_n^h\|_\infty^2$. Suppose that $h \geq 0$, we are interested in the following quantity

$$A_{n,\alpha} = \mathbb{E}_P[\|Y_n^h\|_\infty^2 I_{\{\|Y_n^h\|_\infty^2 \geq \alpha\}}].$$

From Fubini theorem it follows

$$A_{n,\alpha} = \alpha P(\|Y_n^h\|_\infty^2 \geq \alpha) + \int_\alpha^\infty P(\|Y_n^h\|_\infty^2 \geq t)dt.$$

Since $h \geq 0$, the paths of Y_n^h are **piecewise monotonic** thus

$$\|Y_n^h\|_\infty = \sqrt{n} \underset{1\leq k\leq n}{Max} \left| \sum_{j=1}^k U_j \int_{\frac{j-1}{n}}^{\frac{j}{n}} h(s)ds \right|.$$

We have to control the term

$$\mathbf{P}(\|\mathbf{Y_n^h}\|_\infty^{\mathbf{2}} \geq \mathbf{t}) \quad \text{with} \quad \|\mathbf{Y_n^h}\|_\infty = \sqrt{n}\underset{\mathbf{1 \leq k \leq n}}{\mathbf{Max}}\left|\sum_{\mathbf{j=1}}^{\mathbf{k}} \mathbf{U_j} \int_{\frac{\mathbf{j-1}}{\mathbf{n}}}^{\frac{\mathbf{j}}{\mathbf{n}}} \mathbf{h(s)ds}\right|.$$

Lemma : (Billingsley p.69) Let $(\xi_1, \ldots, \xi_n)$ be independent random variables with variances $(\sigma_1^2, \ldots, \sigma_n^2)$. Denote $\forall 1 \leq i \leq n$, $s_i^2 = \sigma_1^2 + \ldots + \sigma_i^2$ and $S_i = \xi_1 + \ldots + \xi_i$, thus $\forall \lambda \in \mathbb{R}$,

$$P\left(\underset{1 \leq i \leq n}{Max}|S_i| \geq \lambda s_n\right) \leq 2P(|S_n| \geq (\lambda - \sqrt{2})s_n)..$$

Thus

$$P(\|Y_n^h\|_\infty^2 \geq t) \leq 2P\left(\left|Y_n^h(1)\right| \geq \sqrt{t} - \sqrt{2s_n^2}\right)$$

where $s_n^2 = n \sum_{k=1}^{n} \left[\int_{\frac{k-1}{n}}^{\frac{k}{n}} h(s)ds\right]^2 \underset{n \to \infty}{\longrightarrow} \int_0^1 h^2(s)d(s)$.

$$A_{n,\alpha} \leq \underbrace{2\alpha P\left(|Y_n^h(1)| \geq \frac{\sqrt{\alpha}}{2} - \sqrt{2s_n^2}\right)}_{*} + \underbrace{2\mathbb{E}_P\left[\left(\left[Y_n^h(1) + \sqrt{2s_n^2}\right]^2 - \alpha\right)_+\right]}_{**}.$$

$*$ According to Dini theorem, if N is a standard normal variable

$$\alpha P\left(|Y_n^h(1)| \geq \frac{\sqrt{\alpha}}{2} - \sqrt{2s_n^2}\right) \underset{n\to\infty}{\longrightarrow} \alpha P\left(|N| \geq \frac{\sqrt{\alpha}}{2\sqrt{\int_0^1 h^2(s)d(s)}} - \sqrt{2}\right).$$

$**$ Moreover, by independence of the U_k, the $\left(Y_n^h(1)\right)^2$ are uniformly integrable hence

$$\mathbb{E}_P\left[\left(\left[Y_n^h(1) + \sqrt{2s_n^2}\right]^2 - \alpha\right)_+\right] \underset{n\to\infty}{\longrightarrow} \mathbb{E}_P\left[\left(\int_0^1 h^2(s)ds\ [N + \sqrt{2}]^2 - \alpha\right)_+\right]$$

and

$$\underset{n}{lim\ sup}\ A_{n,\alpha} \underset{\alpha\to\infty}{\longrightarrow} 0.\square$$

B] Approximation of Gaussian processes

Let $(Y^K(t))_{t\in[0,1]}$ be a continuous Gaussian process of the form

$$Y^K(t) = \int_0^1 K(t,s)dB_s,$$

the kernel $K : [0,1]^2 \to \mathbb{R}$ fulfilling the two following hypotheses :

i) K is measurable and $K(0,r) = 0$, $r \in [0,1]$.

ii) There exist a continuous increasing function $G : [0,1] \to \mathbb{R}$ and a constant α such that for all $0 \le t_1 < t_2 \le 1$

$$\int_0^1 (K(t_2,r) - K(t_1,r))^2 dr \le (G(t_2) - G(t_1))^\alpha.$$

Remark : This is the case of the **fractional Brownian motion** with Hurst parameter $0 < H < 1$ or of the **Ornstein-Uhlenbeck process.**

R. Delgado, M. Jolis : Weak approximation for a class of gaussian processes, J. Appl. Prob 37, 400-407, 2000.

We are interested in the convergence in Dirichlet law of the continuous process $\mathbf{Y_n^K = \int_0^1 K(.,s) dX_n(s)}$.

Proposition : If the kernel K fulfills properties $i)$ and $ii)$ and if $(U_1, U_1^{\#}) \in L^p(W) \times L^p(W \otimes \widetilde{W})$ with $p > \frac{2}{\alpha} \vee 2$ then the process Y_n^K converges in Dirichlet law toward $(Y^K)_* S_{OU}$.

Proof :

$$\begin{cases} -\text{Weak convergence of } Y_n^K \\ -\text{Generalized functional calculus} \\ -\text{Uniform integrability of } \| Y_n^K \|_\infty^2 \end{cases}$$

C] Multiple integrals given by a multi-measure

Remark : If h is sufficiently regular (with bounded variations and right continuous),

$$\mathbf{X_n} \overset{\mathbb{D}-\text{law}}{\longrightarrow} \mathbf{S_{OU}} \Rightarrow \int_0^{\cdot} \mathbf{h(s)dX_n(s)} \overset{\mathbb{D}-\text{law}}{\longrightarrow} (\int_0^{\cdot} \mathbf{h(s)dB_s})_* \mathbf{S_{OU}}.$$

I.P.P gives,

$$\int_0^t h(s)dX_n(s) = \int_0^t X_n(s)d\bar{\nu}_t(s) = \phi_h(X_n)$$

where $\phi_h : \mathcal{C} \to \mathcal{C}$ is $\mathbf{C^1}$ **and lipschitz**.

Thus $\phi_\mathbf{h}(\mathbf{X_n})$ converges in Dirichlet law towards $\phi_{\mathbf{h}\ *}\mathbf{S_{OU}}$.

From Ito formula

$$\int_0^t h(s)dB_s = \int_0^t B_s d\bar{\nu}_t(s) \text{ thus } \phi_{h\ *}S_{OU} = \left(\int_0^{\cdot} h(s)dB_s\right)_{*}S_{OU}.$$

We study the convergence in Dirichlet law of

$$\int_{[0,t]^p} \mathbf{h}(\mathbf{x_1}, \ldots, \mathbf{x_p}) \mathbf{d}\mathbf{X_n}(\mathbf{x_1}) \ldots \mathbf{d}\mathbf{X_n}(\mathbf{x_p}) \qquad (*)$$

when h is regular.

The weak convergence of (*) was proved by *Bardina and Jolis* studying the question of the continuous extension on the Wiener space of the functional

$$\phi_h : \eta \in C.M \mapsto \int_{[0,.]^p} h(x_1, \ldots, x_p) d\eta(x_1) \ldots d\eta(x_p) \in \mathcal{C},$$

X. Bardina, M. Jolis : Weak convergence to the multiple Stratonovich integral, Sto. Proc. Appl, 90, 277-300, 2000.

We need the notion of multimeasure.

Definition : A mapping $\nu : (\mathcal{B}([0,1]))^p \to \mathbb{R}$ is said to be a multimeasure if $\forall i \in \{1, \ldots, p\}$, $\forall (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_p) \in (\mathcal{B}([0,1]))^{p-1}$, the function

$$A \in \mathcal{B}([0,1]) \mapsto \nu(A_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_p)$$

is a signed measure. Moreover we say that h is given by a multimeasure ν if

$$h(x_1, \ldots, x_p) = \nu([x_1, 1], \ldots, [x_p, 1]).$$

Proposition : *(Bardina-Jolis)* The following statements are equivalent :

a) ϕ_h possesses a continuous extension on $\mathcal{C}$.

b) The function h is given by a multimeasure ν.

Moreover, the extension of ϕ_h is such that $\forall \eta \in \mathcal{C}$

$$\phi_h(\eta) = \int_{[0,.]^p} \eta(x_1)\dots\eta(x_p)d\beta_.(x_1,\dots,x_n)$$

where $(\beta_t)_{t\in[0,1]}$ is a family of multimeasure that satisfies

$\|\beta_t\|_{FV} \leq \|\nu\|_{FV}$ ($\|\|_{FV}$ being the Frechet variation.)

Thus, **when $\mathbf{h}$ is given by a multimeasure**

$$\int_{[0,t]^p} \mathbf{h(x_1,\dots,x_p)dX_n(x_1)\dots dX_n(x_p) = \phi_h(X_n)}$$

where $\phi_h : \mathcal{C} \to \mathcal{C}$ is of **classe $\mathbf{C^1}$** and fulfills

$$|\phi_\mathbf{h}(\mathbf{x})| \leq \|\nu\|_\mathbf{FV}\ \|\mathbf{x}\|_\infty^\mathbf{p},$$

$$|\phi'_\mathbf{h}(\mathbf{x})[\tilde{\mathbf{x}}]| \leq \mathbf{p}\ \|\nu\|_\mathbf{FV}\ \|\tilde{\mathbf{x}}\|_\infty\ \|\mathbf{x}\|_\infty^{\mathbf{p-1}}.$$

First consequence : $\phi_h(X_n)$ weakly converges in $\mathcal{C}$ towards $\phi_h(B)$.

Thus it remains to show two points :

• $\phi_h(X_n) \in \mathbb{D}_{\mathcal{C}}$?

•• Study the convergence of $\mathcal{E}[F(\phi_h(X_n))]$, $\forall F \in C^1 \cap Lip$ and identify the limit.

• For the first point we have the following extension of the functional calculus

Proposition : Let $S = (W, \mathcal{W}, P, \mathbb{D}, \Gamma)$ satisfying (G) and $X \in \mathbb{D}_B$. $\forall F \in C^1(B, \mathbb{R})$ such that $|F(x)| \leq K\|x\|_B^p$ and $\|F'(x)\| \leq K\|x\|_B^{p-1}$ one has

$$\left\{\begin{array}{l} \|X\|_B \in L^{2p}(W) \\ \\ \|X^\#\|_B \in L^{2p}(W \otimes \tilde{W}) \end{array}\right. \Longrightarrow \left\{\begin{array}{l} F(X) \in \mathbb{D} \\ \\ F(X)^\# = F'(X)[X^\#] \end{array}\right.$$

Corollary : $\left\{\begin{array}{l} -\text{If } \ (U_1, U_1^\#) \in L^{2p}(W) \times L^{2p}(W \otimes \tilde{W}) \text{ thus } \phi_h(X_n) \in \mathbb{D}_{\mathcal{C}} \\ \\ -\phi_h \in (\mathbb{D}_{OU})_{\mathcal{C}} \end{array}\right.$

Supposing that $(\mathbf{U}_1, \mathbf{U}_1^\#) \in \mathbf{L}^{2p}(\mathbf{W}) \times \mathbf{L}^{2p}(\mathbf{W} \otimes \tilde{\mathbf{W}})$, we can study the convergence in Dirichlet law of $\phi_h(X_n)$. The limit will be $(\phi_h)_* S_{OU}$.

•• Let $F \in C^1(\mathcal{C},\mathbb{R}) \cap Lip$, it follows from the preceding result that

$$\mathcal{E}[F(\phi_h(X_n))] = \int_W \int_{\tilde{W}} \Phi(X_n, X_n^{\#}) dP d\tilde{P}$$

where $\Phi : \mathcal{C}^2 \to \mathbb{R}$ is continuous and such that

$$|\Phi(x,y)| \leq K\|x\|_\infty^{2(p-1)}\|y\|_\infty^2 \leq K max(\|x\|_\infty, \|y\|_\infty)^{2p}.$$

We have the following lemma

Lemma : Let $\Phi : \mathcal{C} \to \mathbb{R}$ be a continuous function such that $\forall x \in \mathcal{C}$, $|\Phi(x)| \leq K(1 + \|x\|_\infty^q)$. When $U_1 \in L^q(W)$

$$\mathbb{E}_P[\Phi(X_n)] \underset{n\to\infty}{\to} \mathbb{E}_\mu[\Phi].$$

thus

$$\mathcal{E}[F(\phi_h(X_n))] \underset{n\to\infty}{\to} \mathcal{E}_{OU}[F(\phi_h)].$$

Proposition : If $h : \mathbb{R}^p \to \mathbb{R}$ is given by a multimeasure and if $(U_1, U_1^{\#}) \in L^{2p}(W) \times L^{2p}(W \otimes \hat{W})$ then $\phi_h(X_n)$ converges in Dirichlet law toward $\phi_h\ {}_*S_{OU}$.

Concluding remarks

• We can show that the preceding results extend in the framework of the functional Lindeberg-Feller theorem.

• Ornstein-Uhlenbeck structure on the Wiener space as a limit object.

• Generalization of this results for SDE ? ? ?