Probability

4. Discrete random variables

Exercises: Uniform Random Variables

Exercise 1 Let X and Y be uniform random variables on $\{0, 1, ..., n\}$. Suppose X and Y are independant. Find $P(X = Y)$ and $P(X \leq Y)$.

Exercises: Poisson law

Exercise 2 Suppose that $X \hookrightarrow \mathcal{P}(\lambda)$. Compute $E[X]$ and $V[X]$.

Exercise 3 Let X (resp. Y) be a random variable following a $\mathcal{P}(\lambda_1)$ (resp. $\mathcal{P}(\lambda_2)$) distribution. Suppose that X and Y are independant. Show that $X+Y \hookrightarrow \mathcal{P}(\lambda_1+\lambda_2)$.

Exercise 4 Suppose that $X \hookrightarrow \mathcal{P}(\lambda)$ and $Y \hookrightarrow \mathcal{P}(\mu)$. For $n \in \mathbb{N}$, find the distribution of X given $X + Y = n$ if X and Y are independant.

Exercise 5 Suppose that $X \hookrightarrow \mathcal{P}(\lambda)$. Let us define the random variable Y such that

$$Y = 0 \text{ if } X \text{ is odd and } Y = n \text{ if } X = 2n\ .$$

Find the distribution of Y and compute $E[Y]$

Exercise 6 Suppose that $X \hookrightarrow \mathcal{P}(\lambda)$, show that

$$P(X \text{ is even}) > P(X \text{ is odd})$$

Exercise 7 Let X be a poisson random variable $\mathcal{P}(\lambda)$.

1) Find $E(\frac{1}{1+X})$.

2) Let $Y = (-1)^X$. Find the distribution probability of Y. Find $E(Y)$ and $Var(Y)$.

Exercise 8 (Poisson approximation to the Binomial) Let P be a Binomial probability with probability of success p and number of trials n. Let $\lambda = pn$. Show that

$$P(k \text{ successes}) = \frac{\lambda^k}{k!}(1 - \frac{\lambda}{n})^n[(\frac{n}{n})(\frac{n-1}{n})...(\frac{n-k+1}{n})](1 - \frac{\lambda}{n})^{-k}.$$

Let $p \to \infty$ ane let p change so that λ remains constant. Conclude that for small p and large n,

$$P(k \text{ successes}) \approx \frac{\lambda^k}{k!} e^{-\lambda}.$$

Exercise 9 Let $(X_1, ..., X_n)$ be i.i.d random variables distributed according a $\mathcal{P}(\lambda)$. Prove that

$$P\left(\left|\frac{X_1 + ... + X_n}{n} - \lambda\right| \geq \varepsilon\right) \leq \frac{\lambda}{n\varepsilon^2}.$$

Exercise 10 Consider a toll gate with m counters. Let N be the number of cars arriving in an hour. We suppose that $N \hookrightarrow \mathcal{P}(\lambda)$. The drivers choose randomly and independently a counter. We denote by X_i the number of cars wich passed at the counter i.

a) Compute $P(X_i = k|N = n)$.

b) Find $E[X_i]$ and $V(X_i)$.

Exercises: Geometric random variables

Exercise 11 Find the expectation and variance of a geometric random variable.

Exercise 12 Let $X \hookrightarrow \mathcal{G}(p)$, $p \in]0, 1[$. Prove that $\forall (k, k_0) \in (\mathbb{N})^2$

$$P(X \geq k_0 + k|X > k_0) = P(X \geq k).$$

Exercise 13 There are a white balls and b black balls in a box. We draw balls one by one (with replacement). Let X_1 be the rank of the first draw of a white ball and X_2 be the rank of the second draw of a white ball.

a) Find the distribution, the expectation and the variance of X_1.

b) Find the distribution and the expectation of X_2.

c) Compare $E[X_1]$ and $E[X_2]$.

Exercise 14 Let X be a Geometrical random variable $\mathcal{G}(b)$, with $0 < b < 1$. Let $m \in \mathbb{N}^*$. Let

$$Y = \max\{X, m\}$$

and

$$Z = \min\{X, m\}.$$

Find the probability distribution of Y. Prove that $Y + Z = X + m$. Find $E(Y)$ and $E(Z)$.

Exercises: general discrete random variables

Exercise 15 Let X be a discrete random variable. Prove that for every $a \in \mathbb{R}$,

$$E((X - a)^2) = Var(X) + (E(X) - a)^2.$$

Deduce from this result the infimum of the mapping $a \to E((X-a)^2)$.

Exercise 16 Let $I_n = \{\frac{k}{n};\ k \in \{1, ..., n\}\}$. Let $X_n : \Omega \to I_n$ the random variable such that $P(X_n = \frac{k}{n}) = \frac{a_n k}{n^2+k^2}$.

a) Find the limit of the sequence (a_n).

b) Prove that $lim E[X_n] = \frac{4-\pi}{2log(2)}$.

Exercise 17 Let $X : \Omega \to \mathbb{N}$.

I) a) Prove that

$$\sum_{k=0}^{n} P(X > k) = \sum_{k=1}^{n} kP(X = k) + (n+1)P(X > n).$$

b) Deduce that $\sum_{k=0}^{\infty} P(X > k) < \infty$ implies $E(|X|) < \infty$.

II) Suppose that $E(|X|) < \infty$.
a) Show that

$$(n+1)P(X > n) \leq \sum_{k=n+1}^{\infty} kP(X = k).$$

b) Show that $\sum_{k=0}^{\infty} P(X > k) < \infty$ and that $E(X) = \sum_{k=0}^{\infty} P(X > k)$.

III) A mark of detergent publish 4 different collector cards in their packages. A housewife buys the detergent in order to offer the cards to her son.

a) Compute P(after n purchases, at least one card is missing).

b) We denote by X the number of packages bought to obtain the four cards for the first time. Compute $E[X]$.

Exercise 18 There are n white balls in a box (balls have numbers $1, 2, ..., n$) and two black balls (in the same box) with number 1 and 2. We draw balls one by one (without replacement). Let X be the rank of the first draw of a white ball, and let Y be the rank of the first draw of a ball with number one.
a) Give the distribution of X and Y.
b) are X and Y independant ?

Exercise 19 Let X_1,...,X_n be independant random variables with same distributions given by:
$P(X_i = 0) = P(X_i = 2) = \frac{1}{4}$ and $P(X_i = 1) = \frac{1}{2}$.
Let $S_n = X_1 + ... + X_n$.
a) Find $E(S_n)$ and $V(S_n)$.
b) Give a necessary and sufficient condition on $n \in \mathbb{N}$ to have

$$P(\frac{1}{2} \leq \frac{S_n}{n} \leq \frac{3}{2}) \geq 0,999).$$