Probability

Conditional expectation

Exercise 1 Let $(X_n)_{n\in\mathbb{N}^*}$ be a sequence of i.i.d random variables such that $P(X_n = -1) = 1-p$ and $P(X_n = 1) = p$ with $p \in]0,1[$. For $n \geq 2$, we define $Y_n = X_{n-1}X_n$.

a) Find the distributions of (Y_n) and of (Y_n, Y_{n+1}).

b) Determine $E[Y_{n+1}|Y_n]$ and find its distribution.

Exercise 2 Let (X,Y) be a pair of random variables having the density (with respect to the Lebesgue measure on $\mathbb{R}^2$)

$$f(x,y) = \frac{1}{\sqrt{2\pi}} x e^{-\frac{x(y+x)}{2}} 1_D(x,y)$$

with $D = \{(x,y)/x > 0, y > 0\}$.

Find the conditional density of Y given $X = x$. Compute $E[Y|X]$.

Exercise 3 Let (X,Y) be a pair of random variables having the density (with respect to the Lebesgue measure on $\mathbb{R}^2$)

$$f(x,y) = \frac{1}{4\pi} e^{-\frac{1}{2}(\frac{x^2}{2} - xy + y^2)}.$$

a) Find the conditional density of Y given $X = x$. Compute $E[Y|X]$ and $E[Y^2|X]$.

b) Prove that there exists a random variable U independent from X such that $Y = \frac{X}{2} + U$. Find the distribution of U.

Exercise 4 Let X and Y be two random variables. Consider the two following propositions:

i) X and Y are independent.

ii) $\forall x$, $E[Y|X = x]$ is independent of x.

a) Is i) $\Rightarrow$ ii)?

b) Is ii) $\Rightarrow$ i)?

Exercise 5 Let (X, Y) be a pair of random variables having the density (with respect to the Lebesgue measure on $\mathbb{R}^2$)

$$f(x,y) = e^{-y} 1_D(x,y)$$

with $D = \{(x,y)/y > x > 0\}$.

a) Find the conditional density of X given $Y = y$.

b) Let $\phi : \mathbb{R} \to \mathbb{R}$ be measurable and bounded. Find $E[\phi(\frac{X}{Y})|Y]$.

c) Show that Y and $\frac{X}{Y}$ are independent (without computing the distribution of $(Y, \frac{X}{Y})$.)