

## Probability

### 3. $\sigma$ -algebra, independence, probability (finite case) and conditional probability, random variables

#### $\sigma$ -algebra

**Exercise 1** Let  $f$  be a function mapping  $\Omega$  to another space  $E$  with a  $\sigma$ -algebra  $\mathcal{E}$ . Let

$$\mathcal{A} = \{A \subset \Omega, \exists B \in \mathcal{E}, A = f^{-1}(B)\}.$$

Show that  $\mathcal{A}$  is a  $\sigma$ -algebra on  $\Omega$ .

**Exercise 2** Let  $\Omega$  be a infinite sample space (countable or not), and let  $\mathcal{A}$  be the family of all subsets of  $\Omega$  which are finite or have a finite complement. Show that  $\mathcal{A}$  is an algebra, but not a  $\sigma$ -algebra.

**Exercise 3** Let  $\mathcal{A}$  be a  $\sigma$ -algebra on a space  $\Omega$  and  $B \in \mathcal{A}$ . Show that  $\mathcal{F} = \{A \cap B; A \in \mathcal{A}\}$  is a  $\sigma$ -algebra of subsets of  $B$ . Is it still true when  $B$  is a subset of  $\Omega$  that does not belong to  $\mathcal{A}$ ?

**Exercise 4** Let  $(\mathcal{G}_\alpha)_{\alpha \in A}$  a family of  $\sigma$ -algebras defined on a space  $\Omega$ . Show that  $\bigcap_{\alpha \in A} \mathcal{G}_\alpha$  is also a  $\sigma$ -algebra.

#### Independence

**Exercise 5** We flip a fair coin twice. Let us consider the three following events :  $A_1 = \{\text{head on first toss}\}$ ,  $A_2 = \{\text{head on second toss}\}$  and  $A_3 = \{\text{head on exactly one toss}\}$ . Show that  $A_1$ ,  $A_2$  and  $A_3$  are pairwise independent but not independent.

**Exercise 5'** Show that if  $A \cap B = \emptyset$  then  $A$  and  $B$  cannot be independent unless  $P(A) = 0$  or  $P(B) = 0$ .

#### Probability

**Exercise 6** Let  $\Omega$  be a finite sample space,  $\mathcal{A}$  be a  $\sigma$ -algebra on  $\Omega$ , and  $P$  be a probability defined on  $\Omega$ . Let  $A$  and  $B$  in  $\mathcal{A}$  such that  $P(A) = \frac{3}{4}$  and  $P(B) = \frac{1}{3}$ . Show that

$$\frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3}.$$

**Exercise 7** A box has  $r$  red and  $b$  black balls. A ball is chosen at random from the box (so that each ball is equally likely to be chosen), and then a second ball is drawn at random from the remaining balls in the box. Find the probabilities that

- a) Both balls are red.
- b) The first ball is red and the second is black.

**Exercise 8** A urn contains  $N$  balls (balls have numbers  $1, \dots, N$ ). We choose  $n$  balls (with replacement) at random. Let  $X$  be the greatest number obtained and  $Y$  the smallest.

- a) Find  $P(X \leq x)$  for  $x \in \{1, \dots, N\}$  and deduce the distribution of  $X$ .
- b) Find  $P(Y \geq x)$  for  $x \in \{1, \dots, N\}$  and deduce the distribution of  $Y$ .

**Exercise 9** A urn contains  $r$  red balls and  $b$  blue balls. A ball is chosen at random from the urn, its color is noted, and it is returned together with  $d$  more balls of the same color. This is repeated indefinitely. What is the probability that:

- a) The second ball drawn is blue?
- b) The first ball drawn is blue given that the second ball drawn is blue?
- c) Let  $B_n$  denote the event that the  $n$ -th ball drawn is blue. Show that  $P(B_n) = P(B_1)$  for all  $n \geq 1$ .

### Conditional probability

**Exercise 10** Donated blood is screened for some disease. Suppose that the test has 99% accuracy (meaning that  $P(\text{test positive} \mid \text{you are ill}) = 0,99$ ), and that one in then thousand people in your age group are ill. The test has a 5% false positive rating, as well. Suppose the test screens you as positive. What is the probability you are ill? Is it 99%?

**Exercise 11** Say we want to do a survey of undergraduate students at the colegio. It is known that 35% of the students are freshmen, 26% are sophomores, 22% are juniors and 17% are seniors. 75% of the freshmen say they like to eat in the cafeteria, versus 62% of the sophomores, 55% of the juniors and 40% of the seniors. If we randomly choose an undergraduate student who is eating in the cafeteria, what is the probability he/she is a senior?

**Exercise 12** A factory produces light bulbs and has three different production shops. (A, B and C). A ensures 20% of the production, B 30% and C 50%. 5% of the light bulbs produced by A are faulty, 4% of the light bulbs produced by B are faulty and 1% of the light bulbs produced by C are faulty.

- a) Find the probability that a light bulb produced by this factory is faulty.
- b) We randomly choose a produced bulb and remark that it is faulty. Find the probability that this bulb has been produced by  $B$ .

**Exercise 13** I want to plant two types of plants in my garden, 30% type A and 70% type B. Suppose both type will either yield red or blue flowers. We know that  $P(\text{red} | A) = 0.4$  and  $P(\text{red} | B) = 0.3$ .

- a) What is the percentage of red flowers I will get ?
- b) Suppose a red flower is picked randomly in my garden. What is the probability of the flower being type A ?

**Exercise 14** Suppose  $A, B, C$  are independent events and  $P(A \cap B) \neq \emptyset$ . Show

$$P(C | A \cap B) = P(C).$$

**Exercise 15** An insurance company insures an equal number of male and female drivers. In any given year, the probability that a male driver has an accident involving a claim is  $\alpha$ , independently of the other years. The analogous probability for females is  $\beta$ . Assume the insurance company selects a driver at random.

- a) What is the probability the selected driver will make a claim this year ?
- b) What is the probability the selected driver makes a claim in two consecutive years ?
- c) Let  $A_1$  and  $A_2$  be the events that a randomly chosen driver makes a claim in each of the first and second years, respectively. Show that  $P(A_2 | A_1) \geq P(A_1)$ .
- d) Find the probability that a claimant is female.

### Random variables

**Exercise 16** Let  $X$  and  $Y$  be uniform random variables on  $\{0, 1, \dots, n\}$ . Suppose  $X$  and  $Y$  are independent. Find  $P(X = Y)$  and  $P(X \leq Y)$ .

**Exercise 17** Find the expectation and variance of a Bernoulli random variable.

**Exercise 18** Let  $X$  and  $Y$  be two independent Bernoulli random variables with the same distributions.

- a) Find the distribution of  $X + Y$  and of  $X - Y$ .
- b) Are  $X + Y$  and  $X - Y$  independent ?

**Exercise 19** A fair coin is tossed 3 times. Let  $X$  equal 0 or 1 accordingly as a head or a tail occurs on the first toss, and let  $Y$  equal the total number of heads that occurs.

- a) Find the distributions of  $X$  and  $Y$ .
- b) Find the distribution of  $(X, Y)$ .
- c) Determine whether or not  $X$  and  $Y$  are independent.
- d) Compute  $\text{cov}(X, Y)$ .
- e) Find the distribution of  $Z = X + Y$ .

f) Compare  $\text{var}(Z)$  and  $\text{var}(X) + \text{var}(Y)$ .

**Exercise 20** We flip a fair coin  $n$  times. Find the distribution of the random variable  $X$  that is equal to the number of tails obtained. Find  $E[X]$  and  $\text{var}(X)$ .

**Exercise 21** Two players flip a fair coin  $n$  times. What is the probability that they obtain the same number of heads?

**Exercise 22** Let  $X$  and  $Y$  be two independent binomial random variables, of parameter  $(n_1, p)$  and  $(n_2, p)$ . Prove that  $X + Y$  is a binomial random variable of parameter  $(n_1 + n_2, p)$ .

**Exercise 23** Let  $X$  be a finite random variable. Prove that for every  $a \in \mathbb{R}$ ,

$$E((X - a)^2) = \text{Var}(X) + (E(X) - a)^2.$$

Deduce from this result the infimum of the mapping  $a \rightarrow E((X - a)^2)$ .

**Exercise 24** Let  $b \in \mathbb{N}$  such that  $b \geq 2$ .

Let a box with one red ball and one green ball. One makes the following experience:

- a) One first draws a ball.
- b) If it is red, one stops the experience.
- c) If it is green, one drops the green ball in the box, then one counts the number of green balls in the box (let  $n$  this number), and one add new green balls in the box in order to have  $b.n$  green balls in the box, and one starts again in a).

Let  $X_b$  the the random variable defined as follows:

$X_b = 0$  if one neither stops in the experience above, and  $X_b = n$  if one stops at draw  $n$ .

i) Compute

$$P(X_b = n)$$

for  $n > 0$ .

ii) Let  $u_n$  be the probability that after  $n$  draws, the experience is not finished. Prove that  $u_n$  converges when  $n \rightarrow +\infty$ .

**Exercise 25** Let  $n = 8.k$ , where  $k \in \mathbb{N}^*$ . Consider the following game:

a) One asks you to choose two integers  $\alpha$  and  $\beta$  such that

$$1 \leq \alpha < \beta < n$$

b) One draws three integers (independently, and each number with the same probability)  $X_1, X_2$  and  $X_3$  between 1 and  $n$ .

c) You win (in euros)  $X_3$  if  $X_3 > \beta$ ,  $X_2$  if  $(X_3 \leq \beta$  and  $X_2 > \alpha)$ , and  $X_1$  if  $(X_3 \leq \beta$  and  $X_2 \leq \alpha)$ .

Let  $Z_{\alpha, \beta}$  the random variable equal to you winnings.

1) Let  $Y_\alpha$  be the random variable equal to  $X_2$  if  $X_2 > \alpha$  and equal to  $X_1$  if  $X_2 \leq \alpha$ .

- i) Find the distribution of  $Y_\alpha$  and its expectation.
- ii) For what value of  $\alpha \in \{1, \dots, n - 2\}$  is the expectation of  $Y_\alpha$  maximal ? what is this maximal value ?

2) Prove that

$$P(Z_{\alpha,\beta} = k \mid X_3 \leq \beta) = P(Y_\alpha = k).$$

3) Deduce from 2) the expectation of  $Z_{\alpha,\beta}$  (as a mapping of  $\beta$ ,  $n$  and the expectation of  $Y_\alpha$ .)

4) What are the best values of  $\alpha$  and  $\beta$  you should choose ?