Probability

2. Combinatory, Introduction to Probability

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Notations: For $1 \leq k \leq n$, we take the notation C_n^k or $\binom{n}{k}$ for $\frac{n!}{(n-k)!k!}$ and the notation A_n^k or $P(n,k)$ for $\frac{n!}{(n-k)!}$.

Exercise 1 Suppose repetitions are not allowed.

a) Find the number n of three digit numbers that can be formed from the six digits: $2,\ 3,\ 5,\ 6,\ 7,\ 9$.

b) How many of them are even?

c) How many of them exceed 400?

Exercice 2 A class contains 10 students with 6 men and 4 women. Find the number n of ways:

a) A 4-member committee can be selected from the students.

b) A 4-member committee with 2 *men* and 2 women can be selected.

c) The class can elect a president, vice-president, treasurer, and secretary.

Exercice 3 Find the number of ways 9 toys may be divided among 4 childrens if the youngest is to receive 3 toys and each of the others 2 toys.

Exercice 4 A pair of dice is tossed and the two numbers appearing on the top are recorded. Find the number of elements in each of the following events.

a) $A = \{\text{two numbers are equal}\}$

b) $B = \{\text{sum is more than } 10\}$

c) $C = \{5 \text{ appears on first die}\}$

d) $D = \{5 \text{ appears on at least one die}\}$.

Exercise 5
a) For $1 \leq k \leq n$ prove that

$$C_{n+1}^{k} = C_n^{k-1} + C_n^k.$$

b) Let $(a,b) \in \mathbb{R}^2$ show that $\forall n \in \mathbb{N}^*$,

$$(a+b)^n = \sum_{k=0}^{n} C_n^k a^k b^{n-k}.$$

c) Deduce from the preceding question that

$$C_n^0 + C_n^1 + ... + C_n^n = 2^n$$

and

$$C_n^0 - C_n^1 + C_n^2 - C_n^3 + ... + (-1)^n C_n^n = 0.$$

Exercise 6 Compute (for every integer n)

$$0.C_n^0 + 1.C_n^1 + 2.C_n^2 + 3.C_n^3 .. + n.C_n^n$$

and

$$\frac{C_n^0}{1} + \frac{C_n^1}{2} + \frac{C_n^2}{3} + \frac{C_n^3}{4} + .. + \frac{C_n^n}{n+1}.$$

Exercise 7 Let E be a set with n elements, compute $card(\mathcal{P}(E))$.

Exercise 8 Find the number of onto mappings from $E = \{1,2,3,4,5\}$ in $F = \{1,2,3,4\}$. Same question with $E = \{1,...,n\}$ and $F = \{1,...,n-1\}$.

Exercise 9 a) Find the number of onto mappings from $E = \{1,...,n\}$ into $F = \{1,2\}$.

b) Let A_1, A_2, A_3 be three finite sets, show that

$$card(A_1 \cup A_2 \cup A_3) = \sum_{i=1}^{3} card(A_i) - \sum_{1 \leq i < j \leq 3} card(A_i \cap A_j) + card(A_1 \cap A_2 \cap A_3).$$

c) Find the number of onto mappings from $E = \{1,...,n\}$ into $F = \{1,2,3\}$.

Exercise 10 a) Let $A_1, ..., A_n$ be n finite sets. Use mathematical induction to prove that

$$\#(\underset{1 \leq i \leq n}{\cup} A_i) = \sum_{i=1}^{n} \#(A_i) - \sum_{1 \leq i < j \leq n} \#(A_i \cap A_j) + \sum_{1 \leq i < j < k \leq n} \#(A_i \cap A_j \cap A_k) + .. + (-1)^n \#(A_1 \cap .. \cap A_n).$$

b) Find the number of onto mappings from $E = \{1,...,n\}$ into $F = \{1,...,k\}$.

c) Find the number of ways to partition a set of n objects into k groups.

Exercise 11 A single card is drawn from an ordinary deck of 52 cards.

a) Find the probability that the card is a king.

b) Find the probability that the card is a face card.

c) Find the probability that the card is a red card.

d) Find the probability that the card is a red face card.

e) Find the probability that the card is a red card or a face card.

Exercise 12 Suppose that in a deck of 52 cards, two are chosen at random .

a) Find the probability that both are spades.

b) Find the probability that the two cards have not the same color (spade, heart, club, diamond.)

c) Find the probability that the first is a spade and the second an heart.

d) Find the probability that one is a spade and one is an heart.

e) Find the probability that one is a spade and one is an ace.

Exercise 13 Find the probability that n people have distinct birthdays. (Here we ignore leap years and assume that a person's birthday can fall on any day with equal probability).

Exercise 14 Each student in a group of n has to choose k exercices in a list of m exercices.

a) Find the probability that all students choose a fixed combination of k exercices .

b) Find the probability that all students choose the same k exercices .

c) Find the probability that all students do not choose a fixed combination of k exercices .

d) Find the probability that at least one fixed combination of k exercices has been chosen.