Probability

1. Reminder on measure theory II: Convergence theorems

Exercice 1 Let $(\Omega, \mathcal{A})$ be a space equiped with a $\sigma-$algebra and μ be a positive measure on $\mathcal{A}$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of non negative mappings in $L^1(\mu)$ and let $f \in L^1(\mu)$.

1) If $f_n \underset{\text{a.s}}{\rightarrow} f$ and $\int f_n d\mu \underset{n\to\infty}{\rightarrow} \int f d\mu$ show that $(f_n)_{n\in\mathbb{N}}$ converges toward f in $L^1(\mu)$.

2) Considering the following mapping:

$$f_n(x) = \frac{x}{n} e^{-\frac{|x|}{n}};\ \forall x \in \mathbb{R},\ \forall n \in \mathbb{N}^*,$$

show that the non negativity condition is necessary in the preceding question.

Exercice 2 Let $(\Omega, \mathcal{A}, P)$ be a probability space and $f \in L^1(P)$.
Show that $nP(E_n) \underset{n\to\infty}{\rightarrow} 0$ where $E_n = \{| f |> n\}$.

Exercice 3 Let $a \in \mathbb{R}$. Show that

$$\sum_{n=0}^{\infty} \int_0^{+\infty} e^{ax} \frac{x^n}{n!} dx = \int_0^{+\infty} \sum_{n=0}^{\infty} e^{ax} \frac{x^n}{n!} dx$$

where dx is the lebesgue measure on $\mathbb{R}$.

Give a condition on a in such a way that the preceding quantities are finite.

Exercice 4 For $q \in \mathbb{N}$,

1) Show that

$$\lim_{n\to\infty} \int_0^n x^q (1 - \frac{x}{n})^n dx = \int_0^{+\infty} x^q e^{-x} dx.$$

2) Compute

$$\lim_{n\to\infty} \int_0^{+\infty} x^q e^{-x} dx$$