Probability

1. Elementary Set theory, Induction

Exercise 1 Prove the following:

a) $A \subset B$ if and only if $A \cap B^c = \emptyset$.
b) $A \subset B$ if and only if $B^c \subset A^c$.
c) $A \subset B$ if and only if $A \cap B = A$.
d) $A \subset B$ if and only if $A \cup B = B$.
e) A is the dijoint union of $A \backslash B$ and $A \cap B$.

Exercise 2 Let A, B, C be three sets such that

$$A \cup B \subset A \cup C$$

and

$$A \cap B \subset A \cap C.$$

Prove that $B \subset C$.

Exercise 3 Show that we can have $A \cup B = A \cup C$ without $B = C$.

Exercise 4 Show that we can have $A \cap B = A \cap C$ without $B = C$.

Exercise 5 Prove that for any class of set $\{A_i; i \in I\}$ and any set B, one has

$$B \cup (\cap_{i \in I} A_i) = \cap_{i \in I}(B \cup A_i)$$

and

$$B \cap (\cup_{i \in I} A_i) = \cup_{i \in I}(B \cap A_i).$$

Exercice 6 Let (A_n) be a sequence of sets. Let

$$A = \cup_{i \in \mathbb{N}}(\cap_{j \geq i} A_j)$$

and

$$B = \cap_{i \in \mathbb{N}}(\cup_{j \geq i} A_j).$$

Are the relations $A = B$, $A \subset B$, $B \subset A$ true ?

Exercise 7 Let $A = \{a, b\}$. Give explicitly the sets $\mathcal{P}(A)$ and $\mathcal{P}(\mathcal{P}(A))$.

Exercise 8 Let A, B be two sets. The symmetric difference of A and B, denoted $A\Delta B$, is defined by

$$A\Delta B = (A \cup B)\backslash(A \cap B).$$

a) Prove that $A\Delta B = (A\backslash B) \cup (B\backslash A)$.
b) Let $A = [1, 2, 3, 4, 5]$ and $B = [4, 5, 6, 7]$, find $A\Delta B$.
c) For every $A \subset E$, find $X \subset E$ such that $A\Delta X = A$.

Exercise 9 For every subset A of a set E, one defines the mapping $1_A : E \to \mathbb{R}$ by $1_A(x) = 0$ if $x \notin A$, $1_A(x) = 1$ if $x \in A$.

a) Prove that $A \subset B$ if and only if $1_A \leq 1_B$.
b) Prove that $1_A^2 = 1_A$, that $1_{A^c} = 1 - 1_A$ and that $1_{A\cap B} = 1_A.1_B$.
c) Prove that $1_{A\cup B} = 1_A + 1_B - 1_A.1_B$ and that $1_{A\Delta B} = 1_A + 1_B - 2.1_A.1_B$.

Exercise 10 Let A and B be two sets. What is the set $(A \times B) \cap (B \times A)$?

Exercise 11 Let A, B and C be three sets. Prove that $(A \times (B \cap C)) = (A \times B) \cap (A \times C)$?

Exercise 12 Each toss of a coin will yield either a head or a tail. Let $C = \{H, T\}$ denote the set of the outcomes. Find C^3, $n(C^3)$ and explain what C^3 represents.

Exercise 13 Find all partitions of $A = \{a, b, c\}$.

Exercise 14 Find all partitions of $A = \{a, b, c, d\}$.

Exercise 15 Let $S = \{a, b, c, d, e, f, g\}$. Determine wich of the following are partitions of S
a) $P_1 = [\{a, c, e\}, \{b\}, \{d, g\}]$
b) $P_2 = [\{a, e, g\}, \{c, d\}, \{b, e, f\}]$
c) $P_3 = [\{a, b, e, g\}, \{c\}, \{d, f\}]$
d) $P_4 = [\{a, b, c, d, e, f, g\}]$

Exercise 16 Let $\{A_1, ..., A_n\}$ and $\{B_1, ..., B_m\}$ two partitions of a set E. Prove that $\{A_i \cap B_j, i = 1, ..., n, j = 1, ..., m\}$ is a partition of E.

Exercise 17 Among 100 students, 30 learn mathematics, 40 learn french, and 10 learn both mathematics and french. Find the number of students who:

i) Do not learn Mathematics.
ii) Learn Mathematics and French.
iii) Learn Mathematics but not French.
iv) Learn French but not Mathematics.
v) Learn exactly one of the two subjetcs.
vi) Learn neither French nor Mathematics.

Exercice 18 Each student at some college has a mathematics requirement M (to take at least one mathematics course) and a science requirement S (to take at least one science course). A poll of 140 sophomore students shows that 60 completed M, 45 completed S, 20 completed both M and S. Find the number of students who had completed

a) At least one of the two requirements.
b) Exactly one of the two requirements.
c) Neither requirement.

Exercise 19 In a survey of 60 people, it was found that 25 read Newsweek magazine, 26 read Time and 23 read Fortune. Also 9 read both Newsweek and Fortune, 11 read both Newsweek and Time, 8 reab both Time and Fortune and 3 read all the three magazines. Find the number of people who read

a) Only Newsweek.
b) Only Time.
c) Only Fortune.
d) Newsweek and Time but not Fortune.
e) Only one of the magazines.
f) None of the magazines.

Exercise 20 Prove the following assertion for $n \geq 0$,

$$A_n : 1 + 2 + 3 + 3... + n = \frac{1}{2}n(n+1).$$

Exercise 21 Prove the following assertion for $n \geq 0$,

$$A_n : 1 + 2 + 2^2 + ... + 2^n = 2^{n+1} - 1.$$

Exercise 22 Prove the following assertion for $n \geq 0$,

$$A_n : 1 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Exercise 23 Prove the following assertion for $n \geq 0$,

$$A_n : \frac{1}{1.3} + \frac{1}{3.5} + ... + \frac{1}{(2n-1)(2n+1)} = \frac{1}{2n+1}.$$

Exercise 24 Suppose a newly-born pair of rabbits, one male, one female, are put in a field. Rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits. Suppose that our rabbits never die and that the female always produces one new pair (one male, one female) every month from the second month on. If we denote by F_n the number of pairs of rabbits after n months prove that

a) For $n \geq 1$, $F_{n+2} = F_{n+1} + F_n$.
b) For $n \geq 0$, $F_n F_{n+2} = F_{n+1}^2 + (-1)^n$.
c) We set $a = \frac{1+\sqrt{5}}{2}$. Show that $a^2 = a + 1$ and that for $n \geq 0$, $F_{n+1} \leq a^{n-1}$.

Exercise 25 Let us consider the following proposition

$A(n)$: "In a group of n students, if at least one of them is a girl, all are girls".

a) Prove that $A(n) \Rightarrow A(n+1)$ if $n \geq 2$.
b) Is $A(n)$ true for all $n \geq 1$?