Probability

1. Reminder on measure theory I

Exercice 0 1. Let $(X_n : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})))_{n\in\mathbb{N}}$ be a sequence of measurable functions

a) Show that $\underset{n\in\mathbb{N}}{sup}\ X_n$ and $\underset{n\in\mathbb{N}}{inf}\ X_n$ are measurable

b) Define

$$liminf\ X_n = \underset{n\in\mathbb{N}}{sup}\ \underset{k\geq n}{inf}\ X_k$$

$$limsup\ X_n = \underset{n\in\mathbb{N}}{inf}\ \underset{k\geq n}{sup}\ X_k.$$

Show that $liminf\ X_n$ and $limsup\ X_n$ are measurable.

2. Let $X : (\Omega, \mathcal{A}) \to (E, \mathcal{E})$ and $Y : (E, \mathcal{E}) \to (F, \mathcal{F})$ be two random variables. Show that $Y \circ X$ is measurable.

3. Let $X : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $Y : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be two random variables.

a) Show that (X, Y) is measurable.

b) Show that $X + Y$, XY and $\frac{X}{Y}$ (If $Y > 0$) are measurable.

Exercice 1 1. Consider a sample space Ω. Find $\sigma(\mathcal{C})$ when

a) $\mathcal{C} = \{A\}$, $A \subset \Omega$

b) $\mathcal{C} = \{B \in \mathcal{P}(\Omega);\ B \subset A\}$, $A \subset \Omega$

2. Let $f : (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be defined by $f(x, y) = x^2 + y^2$. Find $\sigma(f)$.

Exercice 2 Suppose that $\mathcal{C} \subset \mathcal{P}(\Omega)$ is closed under finite intersection. Let Q_1 and Q_2 be two measures on $\sigma(\mathcal{C})$ such that $Q_1 = Q_2$ on $\mathcal{C}$.

a) If Q_1 and Q_2 are probability measures show that $Q_1 = Q_2$ on $\sigma(\mathcal{C})$.

b) If there exists an increasing sequence $(C_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{C}$ such that $\Omega = \cup_{n\in\mathbb{N}} C_n$ and $\forall n \in \mathbb{N}$, $Q_1(C_n) = Q_2(C_n) < \infty$, show that $Q_1 = Q_2$ on $\sigma(\mathcal{C})$.

Exercice 3 Let (A_n) be a sequence of sets. We define

$$liminf\ A_n = \cup_{i\in\mathbb{N}}(\cap_{j\geq i} A_j)$$

and

$$limsup\ A_n = \cap_{i\in\mathbb{N}}(\cup_{j\geq i} A_j).$$

1) Prove the following relations

$$(liminf\ A_n)^c = limsup\ A_n^c,$$

$$limsup\ A_n \cup B_n = limsup\ A_n \cup limsup\ B_n$$

$$liminf\ A_n \cap limsup\ B_n \subset limsup\ A_n \cap B_n \subset limsup\ A_n \cap limsup\ B_n.$$

2) Find $liminf\ A_n$ and $limsup\ A_n$ in the following cases:

a) $\forall n \in \mathbb{N}$, $A_{2n} = [-1, 2 + \frac{1}{n}[$ and $A_{2n+1} =]-2-\frac{1}{n}, 1]$.

b) Let E and F be two subsets of a sample space Ω, $\forall n \in \mathbb{N}$, $A_{2n} = E$ and $A_{2n+1} = F$.

3) Remind and prove the Borel Cantelli Lemma.