Variance reduction techniques: Around B&S

Christophe Chorro (christophe.chorro@gmail.com)

University Paris 1

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Study Plan

- Variance reduction techniques
  - Introduction
  - Reminder on B&S Model
  - Control Variate
  - Antithetic Variables
  - Importance sampling
  - Conditioning

Biblio: • Stochastic Calculus and Black Scholes:

http://christophe.chorro.fr/docs/CSangl.pdf

• A.G.Z Kemna and A.C.F. Vorst, *A pricing method for options based on average asset values*, J. Banking Finan., 1990, March, 113–129.

Plan

1. Variance reduction techniques
   - Variance reduction techniques:Introduction
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   - Control Variate
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Variance reduction techniques: Introduction

Let $(X_n)_{n\in\mathbb{N}}$ be i.i.d random variables with values in $\mathbb{R}$ such that $\mathbb{E}[|X_1|^2] < \infty$.

For large n, with a confidence of 95%,

$$\mathbb{E}[X_1] \in \left[\frac{S_n}{n} - \frac{1.96\sigma}{\sqrt{n}}, \frac{S_n}{n} + \frac{1.96\sigma}{\sqrt{n}}\right]$$

with $\sigma^2 = Var(X_1)$.

The magnitude of the error is given by $\frac{1.96\sigma}{\sqrt{n}}$: the size of σ is fundamental for the speed of convergence.

Idea: Reduce σ

Find Y such that $\mathbb{E}[X_1] = \mathbb{E}[Y]$ and $Var(X_1) > Var(Y)$.

Plan

Brownian motion

We consider a probability space $(\Omega, \mathcal{A}, P)$.

Definition

Standard Brownian motion (B.M) is a stochastic process $(B_t)_{t\in[0,T]}$ fulfilling :

a) $B_0 = 0$ P-a.s.

b) B is continuous i.e $t \to B_t(w)$ is continuous for P almost all w.

c) B has independent increments: For Si $t > s$, $B_t - B_s$ is independent of $\mathcal{F}_s^{\mathcal{B}} = \sigma(B_u, u \leq s)$.

d) the increments of B are stationary and gaussian: For $t \geq s$, $B_t - B_s$ follows a $\mathcal{N}(0, t-s)$.

Brownian motion

We consider a subdivision $0 = t_0 < ... < t_n = T$ of $[0, T]$. We want to simulate

$$(\mathbf{B_{t_0}}, ...\mathbf{B_{t_n}}).$$

Idea:

$$\mathbf{B_{t_k}} = \mathbf{B_{t_{k-1}}} + \underbrace{\mathbf{B_{t_k}} - \mathbf{B_{t_{k-1}}}}_{\mathcal{N}(\mathbf{0},\mathbf{t_k}-\mathbf{t_{k-1}}) \perp \mathbf{B_{t_{k-1}}},\dots,\mathbf{B_0}} .$$

Proposition

If $(G_1, ...G_n)$ are i.i.d $\mathcal{N}(0,1)$, we define

$$X_0 = 0,\ X_i = \sum_{j=1}^{i} \sqrt{t_j - t_{j-1}} G_j \ i > 0.$$

Then

$$(X_0, ..., X_n) \underset{\mathcal{D}}{=} (B_{t_0}, ...B_{t_n}).$$

Brownian motion

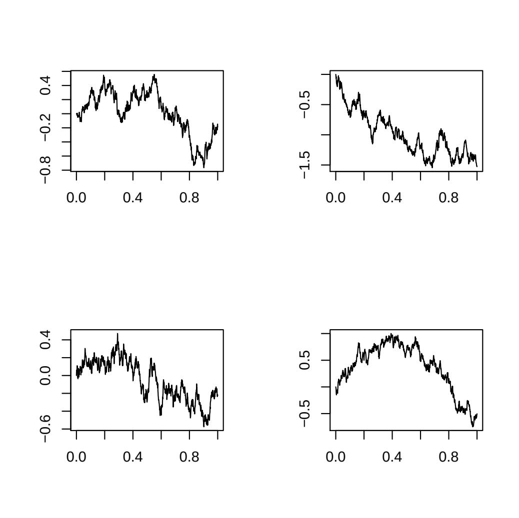

Figure: 4 paths of the Brownian motion on [0, 1] generated using the preceding method with the regular subdivision of step 0.001.

Black Scholes model

We consider the time interval $[0, T]$ and r the risk free rate (supposed to be constant) during this period.

Non-risky asset: Its dynamic is given by

$$S_0^0 = 0, \; S_t^0 = e^{rt}.$$

Risky asset: Under the historical probability P its dynamic is given by the following SDE:

$$\underline{\mathbf{dS_t = \mu S_t dt + \sigma S_t dB_t}} \quad (1)$$

with initial condition $S_0 = x_0 > 0$ and where B is a standard BM under P.

$$\textbf{Itô formula} \quad \Rightarrow \mathbf{S_t = x_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}}.$$

Black Scholes model

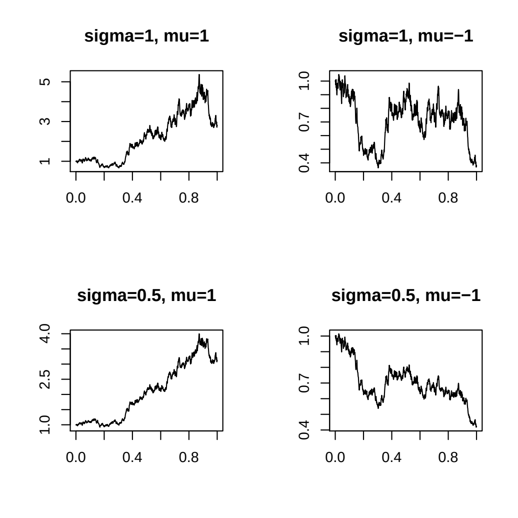

Figure: Simulation of a path of the risky asset in the B&S model for different parameters

Black Scholes model

What is, in this model, the price of a contingent claim with payoff Φ_T at T?

Proposition

In the B&S model there exists a unique probablity $Q \sim P$ such that the price at $t = 0$ of a contingent claim with payoff Φ_T at T is given by

$$price = e^{-rT}\mathbb{E}_Q[\Phi_T].$$

Moreover the dynamic of the risky asset under Q is given by

$$\underline{\mathbf{dS_t = rS_t dt + \sigma S_t dW_t}} \quad (\mu \Leftrightarrow r) \tag{2}$$

where W is a standard BM under Q.

Black Scholes model

Examples: The Black Scholes Formulas

- For Call options ($\Phi_T = (S_T - K)_+$) one has

$$\mathbb{E}_{\mathbf{Q}}[(\mathbf{S_T} - \mathbf{K})_+] = \mathbf{S_0 N(d_1)} - \mathbf{K e^{-rT} N(d_2)} \qquad (3)$$

where

$$d_1 = \frac{log(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \text{ and } d_2 = \frac{log(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \qquad (4)$$

and where N is the distribution function of a $\mathcal{N}(0, 1)$.

- For Put options ($\Phi_T = (K - S_T)_+$) one has

$$\mathbb{E}_{\mathbf{Q}}[(\mathbf{K} - \mathbf{S_T})_+] = -\mathbf{S_0 N(-d_1)} + \mathbf{K e^{-rT} N(-d_2)}. \qquad (5)$$

Black Scholes model

This formulas are fundamental because

- They are easy to compute in practice
- They are a Benchmark to test numerical methods (for example variance reduction techniques)

In the sequel, all the variance reduction methods will be test computing by Monte carlo simulations

$$\mathbb{E}[(K - e^{\sigma G})_+] \text{ or } \mathbb{E}[(e^{\sigma G} - K)_+]$$

where $G \hookrightarrow \mathcal{N}(0,1)$ i.e computing (up to some coefficients) the price of a call

or a put option in a B&S model where $S_0 = 1$ and $r = 0$.

Black Scholes model: Numerical example

- Price of a call $\mathbb{E}[(e^{\sigma G} - K)_+]$ with $\sigma = 0.5$ and $K = 1$
  - Exact value=0.28353
  - Estimated value (N=100) = 0.249, confidence interval at 95% : $[\mathbf{0.164}; \mathbf{0.334}]$
  - Estimated value (N=1000) = 0.279, confidence interval $[\mathbf{0.248}; \mathbf{0.308}]$
  - Estimated value (N=10000) = 0.276, confidence interval $[\mathbf{0.267}; \mathbf{0.285}]$
- Price of a put $\mathbb{E}[(K - e^{\sigma G})_+]$ with $\sigma = 0.5$ and $K = 1$.
  - Exact value=0.15038
  - Estimated value (N=100) = 0.154, confidence interval at 95% : $[\mathbf{0.112}; \mathbf{0.195}]$
  - Estimated value (N=1000) = 0.155, confidence interval $[\mathbf{0.143}; \mathbf{0.167}]$
  - Estimated value (N=10000) = 0.149, confidence interval $[\mathbf{0.145}; \mathbf{0.152}]$

Black Scholes model

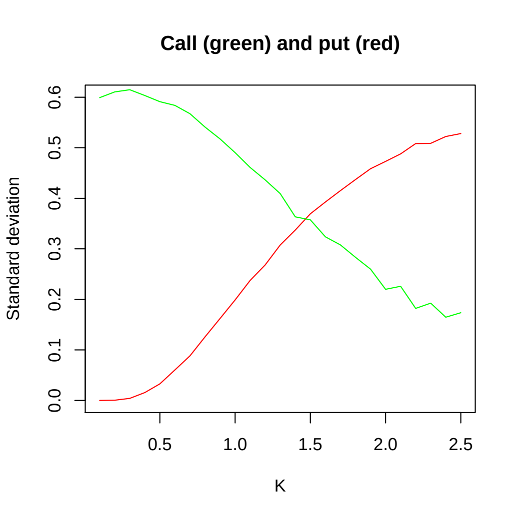

Figure: Standard deviation of the preceding payoffs ($\sigma = 0.5$) for different strikes

Black Scholes model

In practical cases, the moneyness $\frac{S_0}{K}$ of most liquid options (in our example $S_0 = 1$) belongs in general to $[0.7, 1.3]$.

Thus, It's better to price put than call by Monte Carlo methods.

To recover the price of the call we may use the Call-Put parity:

$$\mathbb{E}[(e^{\sigma G} - K)_+] = \mathbb{E}[(-e^{\sigma G} + K)_+] + e^{\frac{\sigma^2}{2}} - K.$$

- Price of a call $\mathbb{E}[(e^{\sigma G} - K)_+]$ with $\sigma = 0.5$ and $K = 1$ computed by Call-Put parity and MC
  - Exact value=0.28353
  - Estimated value (N=100) = 0.281, confidence interval at 95% : [**0.240**; **0.320**]
  - Estimated value (N=1000) = 0.285, confidence interval [**0.272**; **0.297**]
  - Estimated value (N=10000) = 0.287, confidence interval [**0.283**; **0.291**]

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Controle Variate

We want to compute by Monte Carlo method $\mathbb{E}[f(X)]$.

Idea:

$$\mathbb{E}[f(X)] = \mathbb{E}[f(X) - h(X)] + \mathbb{E}[h(X)]$$

where h is chosen to ensure that

- $\mathbb{E}[h(X)]$ may be computed explicitly
- $Var(f(X) - h(X)) \ll Var(f(X))$ (intuitively we look for $f(X) - h(X)$ small).

Example: Call-Put parity

Controle Variate: Basket options (Homework 1)

Aim: **Price by MC method a Basket put in the B&S framework**

Let G_1 and G_2 be two independent $\mathcal{N}(0,1)$ and $(\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}) \in \mathbb{R}^4$. We define

$$S_1 = e^{\sigma_{11} G_1 + \sigma_{12} G_2}$$
$$S_2 = e^{\sigma_{21} G_1 + \sigma_{22} G_2}.$$

For $a_1, a_2 > 0$ with $a_1 + a_2 = 1$, we are interested in the numerical computation (no closed form formula) of

$$E\left[\left(K - \underbrace{(a_1 S_1 + a_2 S_2)}_{X}\right)_+\right].$$

Controle Variate: Basket options (Homework 1)

First Idea: Classical MC methods

With $a_1 = a_2 = 0.5$, $K = 1$, $\sigma_{11} = \sigma_{22} = 0.1$ and $\sigma_{12} = \sigma_{21} = 0.15$,

- Box Muller Method to generate gaussian random variables

Estimated value (N=10000) $=6,334.10^{-2}$, confidence interval at 95% :

$$[\mathbf{6,159.10^{-2};6,508.10^{-2}}]$$

- Inversion Method to generate gaussian random variables:

Estimated value (N=10000) = $6,336.10^{-2}$, confidence interval at 95% :

$$[\mathbf{6,161.10^{-2};6,511.10^{-2}}]$$

Controle Variate: Basket options (Homework 1)

Second Idea: Control variate when $(\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22})$ are small.

We have

$$X \sim e^{Y}$$

where

$$Y = a_1 Log(S_1) + a_2 Log(S_2) \hookrightarrow \mathcal{N}\left(0, \sigma^2\right)$$

with

$$\sigma^2 = (a_1\sigma_{11} + a_2\sigma_{21})^2 + (a_1\sigma_{12} + a_2\sigma_{22})^2$$

And we remark that

$$E\left[(K-X)_+\right] = E\left[(K-X)_+ - \left(K-e^{Y}\right)_+\right] + \mathbb{C}$$

where

$$\mathbb{C} = E\left[\left(K-e^{Y}\right)_+\right] = e^{\frac{\sigma^2}{2}} N(\frac{log(K)}{\sigma} - \sigma) + KN(\frac{log(K)}{\sigma}).$$

Controle Variate: Basket options (Homework 1)

We use a classical Monte Carlo Method ($N = 10000$) to approximate

$$E\left[(K-X)_+ - \left(K-e^Y\right)_+\right]$$

and deduce $E\left[(K-X)_+\right]$.

With $a_1 = a_2 = 0.5$, $K = 1$, $\sigma_{11} = \sigma_{22} = 0.1$ and $\sigma_{12} = \sigma_{21} = 0.15$,

Estimated value (N=10000) $= 6,312.10^{-2}$, confidence interval at 95% :

$$[\mathbf{6,311.10^{-2}}; \mathbf{6,313.10^{-2}}]$$

Controle Variate: Asian Options with Kemna and Vorst Method

Aim: **Price by MC method an asian call in the B&S model**

Thus we have to approximate (no closed form formula)

$$e^{-rT}\,\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_s ds - K\right)_+\right]$$

where $\forall t \in [0, T]$

$$S_t = x_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}.$$

First Idea:

$$\frac{1}{T}\int_0^T S_s ds \approx \frac{1}{p}\sum_{k=0}^{p} S_{\frac{kT}{p}} \; + \; MC$$

With $\sigma = 0.2$, $x_0 = K = 100$, $r = 0.1$, $T = 1$ and $p = 50$ one obtains

- Estimated value (N=10000) = 6.93, confidence interval at 95% : $[\mathbf{6.769}; \mathbf{7.101}]$

Controle Variate: Asian Options with Kemna and Vorst Method

Second Idea: When σ and r are small Kemna and Vorst tell us that

$$\frac{1}{T}\int_0^T S_s ds \approx exp\left(\frac{1}{T}\int_0^T log(S_s)ds\right) = A_T.$$

We are going to use the identity

$$e^{-rT}\,\mathbb{E}\left[\left(\tfrac{1}{T}\int_0^T S_s ds - K\right)_+\right] = \underbrace{e^{-rT}\,\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_s ds - K\right)_+ - (A_T - K)_+\right]}_{(1)}$$

$$+ \underbrace{e^{-rT}\,\mathbb{E}\left[(A_T - K)_+\right]}_{(2)}$$

Controle Variate: Asian Options with Kemna and Vorst Method

For (1) We use the first idea (Approximation by Riemann sums)

For (2) We just have to remark that

$$A_T = exp\left(\frac{1}{T}\int_0^T log(S_s)ds\right) \hookrightarrow exp\left(\mathcal{N}((\widetilde{r} - \frac{\widetilde{\sigma}^2}{2})T, \widetilde{\sigma}^2 T)\right)$$

where $\widetilde{r} = \frac{r}{2} - \frac{\sigma^2}{12}$ and $\widetilde{\sigma} = \frac{\sigma}{\sqrt{3}}$.

Thus we may use Black Scholes formula to compute explicitly (2).

With $\sigma = 0.2$, $x_0 = K = 100$, $r = 0.1$, $T = 1$ and $p = 50$ one obtains

- Estimated value (N=10000) = 6.981, confidence interval at 95% : $[\mathbf{6}.\mathbf{969}; \mathbf{6}.\mathbf{993}]$.

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1. Variance reduction techniques
   - Variance reduction techniques:Introduction
   - Reminder on B&S model
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   - Antithetic Variables
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Antithetic Variables

If $(X_i)_{i\in\mathbb{N}^*}$ are independent with the same distribution than X we may approximate $\mathbb{E}[g(X)]$ by

$$\mathbb{E}[g(X)] \approx \frac{1}{2n}\sum_{k=1}^{2n} g(X_k) = \frac{1}{2n}\sum_{k=1}^{n} g(X_{2k-1}) + g(X_{2k}).$$

Suppose that there exists a transformation $T:\mathbb{R}\to\mathbb{R}$ such that

$T(X)$ has the same distribution than X.

In this case we may use

$$\mathbb{E}[g(X)] = \frac{\mathbb{E}[g(X)+g(T(X))]}{2} \approx \frac{1}{2n}\sum_{k=1}^{n} g(X_k) + g(T(X_k)).$$

Question: Is $Var\,[g(X_1)+g(X_2)] > Var\,[g(X_1)+g(T(X_1))]$?

Antithetic Variables

We have

$$Var\left[g(X_1)+g(X_2)\right]=2Var\left[g(X_1)\right]$$

and

$$Var\left[g(X_1)+g(T(X_1))\right]=2Var\left[g(X_1)\right]+2Cov[g(X_1),g(T(X_1))].$$

So,

$$Cov[g(X_1),g(T(X_1))]<0 \;\Rightarrow\; Var\left[g(X_1)+g(X_2)\right]>Var\left[g(X_1)+g(T(X_1))\right].$$

Under some simple conditions on g and T,

$$Cov[g(X_1),g(T(X_1))]<0 \text{ holds.}$$

Antithetic Variables

Lemma

If X is a random variable and $f, h : \mathbb{R} \to \mathbb{R}$ two non decreasing (or non increasing) mappings,

$$\mathbb{E}[h(X)f(X)] \geq \mathbb{E}[h(X)]\mathbb{E}[f(X)].$$

Corollary

If g is monotonic and T non increasing,

$$Cov[g(X_1), g(T(X_1))] \leq 0.$$

Proof of the lemma: Let Y be a random variable independent from X with the same distribution.

From

$$\mathbb{E}[(h(X)-h(Y))(f(X)-f(Y))]\geq 0$$

we deduce

$$\mathbb{E}[h(X)f(X)]+\mathbb{E}[h(Y)f(Y)]\geq\mathbb{E}[h(X)f(Y)]+\mathbb{E}[h(Y)f(X)].$$

Using independence and equi-distribution

$$\mathbb{E}[h(X)f(X)]\geq\mathbb{E}[h(X)]\mathbb{E}[f(X)]$$

thus

$$Cov[f(X),h(X)]\geq 0.$$

Antithetic Variables for call options in B&S

We want to apply the preceding method to

$$c = \mathbb{E}[(e^{\sigma G} - K)_+].$$

Here we have to remark that

$$g(x) = (e^{\sigma x} - K)_+ \text{ is non decreasing } (\sigma > 0),$$

$$T(x) = -x \text{ is non increasing}$$

$$G \text{ and } -G \text{ have the same distribution,}$$

thus, we may use the following approximation

$$\mathbb{E}[g(G)] \approx \frac{1}{2n}\sum_{k=1}^{n} g(G_k) + g(T(G_k)).$$

Antithetic Variables for call options in B&S

Numerical results for $\sigma = 0.5$ and $K = 1$

- Without antithetic variables
  - Exact value=0.28353
  - Estimated value (n=100) = 0.238, confidence interval at 95% : $[\mathbf{0.151}; \mathbf{0.324}]$
  - Estimated value (n=10000) = 0.285, confidence interval $[\mathbf{0.275}; \mathbf{0.295}]$
- With antithetic variables
  - Exact value=0.28353
  - Estimated value (n=100) = 0.283, confidence interval at 95% : $[\mathbf{0.227}; \mathbf{0.338}]$
  - Estimated value (n=10000) = 0.283, confidence interval $[\mathbf{0.276}; \mathbf{0.289}]$

Plan

1. Variance reduction techniques
   - Variance reduction techniques:Introduction
   - Reminder on B&S model
   - Control Variate
   - Antithetic Variables
   - Importance sampling
   - Conditioning

Importance sampling

Let X having the distribution $f_X(x)dx$, we want to approximate

$$\mathbb{E}[g(X)].$$

We consider another random variable Y with distribution $f_Y(y)dy$. One has

$$\mathbb{E}[g(X)] = \int_{\mathbb{R}} g(x) f_X(x) dx = \int_{\mathbb{R}} g(x) f_X(x) \frac{f_Y(x)}{f_Y(x)} dx = \mathbb{E}\left[g(Y)\frac{f_X(Y)}{f_Y(Y)}\right].$$

If we are able to simulate de distribution of Y we may use MC method to approximate $\mathbb{E}\left[g(Y)\frac{f_X(Y)}{f_Y(Y)}\right]$.

Question: If $Z = g(Y)\frac{f_X(Y)}{f_Y(Y)}$, is $Var(Z) < Var(g(X)$?

Importance sampling

But

$$Var(Z) = \int_{\mathbb{R}} g(x)^2 \frac{f_X^2(x)}{f_Y(x)} dx - \mathbb{E}[g(X)]^2.$$

- When $g \geq 0$, $f_Y(x) = \frac{g(x) f_X(x)}{\mathbb{E}[g(X)]} \Rightarrow Var(Z) = 0$
- Problem: $\mathbb{E}[g(X)]$ is unknown...
- Idea: Take $f_Y(x) \approx \; cste \; \mid \; f_X(x) g(x) \; \mid$.

Importance sampling: example 1

We want to compute by MC methods

$$\int_0^1 cos(\frac{\pi x}{2})dx = \mathbb{E}[cos(\frac{\pi U}{2})] \text{ where} U \hookrightarrow \mathcal{U}([0,1]).$$

- $g(x) = cos(\frac{\pi x}{2})$ is odd, $g(0) = 1$, $g(1) = 0$ thus we take

$$f_Y(x) = cste\ (1 - x^2)\ 1_{[0,1]}(x).$$

- f_Y density $\Rightarrow cste = \frac{3}{2}$
- If Y has the density f_Y,

$$\int_0^1 cos(\frac{\pi x}{2})dx = \mathbb{E}[Z] \text{ where } Z = \frac{2cos(\frac{\pi Y}{2})}{3(1 - Y^2)}.$$

In this case

$$Var(Z) \approx \frac{Var(cos(\frac{\pi U}{2}))}{100}...$$

Importance sampling: example 1

Remark: In the preceding example we are able to simulate Y with density

$$f_Y(x) = \frac{3}{2}\left(1 - x^2\right) 1_{[0,1]}(x)$$

using the next result we have already seen (rejection method...)

Proposition

Let f_Y be a density on $\mathbb{R}$ and

$$\mathcal{D}_{f_Y} = \{(x,u) \in \mathbb{R} \times \mathbb{R}_+ \mid 0 \leq u \leq f_Y(x)\}.$$

Let (Y,U) be a random variable on $\mathbb{R} \times \mathbb{R}_+$, then,

$$(Y,U) \hookrightarrow \mathcal{U}(\mathcal{D}_{f_Y}) \Leftrightarrow Y \text{ has density } f_Y \text{ and } \forall x \in \mathbb{R}_+,\ U \mid x \hookrightarrow \mathcal{U}([0, f_Y(x)]).$$

Importance sampling: Call option in B&S

We want to use the preceding method to compute

$$c = \mathbb{E}[(e^{\sigma G} - 1)_+]$$

where $G \hookrightarrow \mathcal{N}(0,1)$.

Idea: Since σ is small, $e^{\sigma x} - 1 \approx \sigma x$ thus

$$c = \int_{\mathbb{R}_+} (e^{\sigma x} - 1) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \int_{\mathbb{R}_+} \frac{(e^{\sigma x} - 1)}{cste\ \sigma x} \underbrace{\frac{cste\ \sigma x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}_{\approx f_Y(x)} dx$$

$y = x^2$,

$$c = \int_{\mathbb{R}_+} \frac{(e^{\sigma\sqrt{y}} - 1)}{\sqrt{2\pi y}} e^{-\frac{y}{2}} \frac{dx}{2} = \mathbb{E}\left[\frac{(e^{\sigma\sqrt{Y}} - 1)}{\sqrt{2\pi Y}}\right] \text{ where } Y \hookrightarrow \mathcal{E}(0.5).$$

Importance sampling: Call option in B&S

Numerical results for $\sigma = 0.5$:

- Without importance sampling
  - Exact value=0.28353
  - Estimated value (N=100) = 0.249, confidence interval at 95% : $[\mathbf{0.164}; \mathbf{0.334}]$
  - Estimated value (N=10000) = 0.276, confidence interval $[\mathbf{0.267}; \mathbf{0.285}]$
- With importance sampling
  - Exact value=0.28353
  - Estimated value (N=100) = 0.287, confidence interval at 95% : $[\mathbf{0.275}; \mathbf{0.300}]$
  - Estimated value (N=10000) = 0.284, confidence interval $[\mathbf{0.283}; \mathbf{0.285}]$

Plan

1. Variance reduction techniques
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Conditioning

We want to compute by M.C methods

$$\mathbb{E}[g(X,Y)].$$

If $h(X) = \mathbb{E}[g(X,Y) \mid X]$ we have from classical properties that

- $\mathbb{E}[g(X,Y)] = \mathbb{E}[h(X)]$
- From Jensen inequality for conditional expectation:

$$h^2(X) \leq \mathbb{E}[g^2(X,Y) \mid X], \text{so}$$

$$Var[g(X,Y)] \geq Var[h(X)].$$

Pb: MC for $\mathbb{E}[h(X)]$ but we have to know explicitly h.....

Conditioning

Suppose that the pair (X, Y) has the distribution $f_{X,Y}(x,y)dxdy$. We know in this case that:

- The distribution of X is given by $f_X(x)dx$ where $f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,y)dy$.
- The distribution of Y is given by $f_Y(y)dy$ where $f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x,y)dx$.
- The conditional distribution of Y given $X = x$ is given by $f_{Y|X}(x,y)dy$ where
$$f_{Y|X}(x,y) = 1_{f_X(x)>0} \frac{f_{X,Y}(x,y)}{f_X(x)}.$$

Thus

$$h(X) = \mathbb{E}[g(X,Y) \mid X]$$

where

$$h(x) = \int_{\mathbb{R}} g(x,y) f_{Y|X}(x,y)dy.$$

Conditioning: Best of call option

We want to approximate

$$p = \mathbb{E}[(Max(e^{\sigma_1 G_1}, e^{\sigma_2 G_2}) - K)_+]$$

where G_1 and G_2 are two i.i.d $\mathcal{N}(0, 1)$.

First idea: Classical MC...

Second idea: Conditioning by G_1 + MC...

We have $p = \mathbb{E}[h(e^{\sigma_1 G_1})]$ where

$$h(x) = \mathbb{E}[(Max(x, e^{\sigma_2 G_2}) - K)_+]$$

where h may be explicitly known.

Conditioning: Best of call option

In fact $h(x) = \mathbb{E}[\big(Max(x, e^{\sigma_2 G_2}) - K\big)_+]$ but

$$\big(Max(x, e^{\sigma_2 G_2}) - K\big)_+ = (e^{\sigma_2 G_2} - K)_+ \, 1_{\{x \leq K\}} + \big(x - K + (e^{\sigma_2 G_2} - x)_+\big) \, 1_{\{x > K\}}.$$

Thus

$$h(x) = \left(e^{\frac{\sigma_2^2}{2}} N(-\frac{log(K)}{\sigma_2} + \sigma_2) - KN(-\frac{log(K)}{\sigma_2})\right) 1_{\{x \leq K\}} + \left(x - K + e^{\frac{\sigma_2^2}{2}} N(-\frac{log(x)}{\sigma_2} + \sigma_2) - xN(-\frac{log(x)}{\sigma_2})\right) 1_{\{x > K\}}.$$

***h* is explicit.**

Conditioning: Best of call option

With $\sigma_1 = 0.2$, $\sigma_2 = 0.5$, $K = 1$,

- Without conditioning (Exact value 0.338)
  - Estimated value (N=100) = 0.318, confidence interval at 95% : $[\mathbf{0.219}; \mathbf{0.417}]$
  - Estimated value (N=10000) = 0.327, confidence interval at 95% : $[\mathbf{0.318}; \mathbf{0.336}]$
- With conditioning (Exact value 0.338)
  - Estimated value (N=100) = 0.344, confidence interval at 95% : $[\mathbf{0.323}; \mathbf{0.365}]$
  - Estimated value (N=10000) = 0.339, confidence interval at 95% : $[\mathbf{0.338}; \mathbf{0.341}]$