Variance reduction techniques: Around B&S (Lecture 3 part 1)

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Study Plan

- Variance reduction techniques
  - Introduction
  - Reminder on B&S Model
  - Control Variate

Biblio: • Stochastic Calculus and Black Scholes:

http://christophe.chorro.fr/docs/CSangl.pdf

• A.G.Z Kemna and A.C.F. Vorst, *A pricing method for options based on average asset values*, J. Banking Finan., 1990, March, 113–129.

Plan

Variance reduction techniques: Introduction

Let $(X_n)_{n\in\mathbb{N}}$ be i.i.d random variables with values in $\mathbb{R}$ such that $\mathbb{E}[|X_1|^2] < \infty$.

For large n, with a confidence of 95%,

$$\mathbb{E}[X_1] \in \left[\frac{S_n}{n} - \frac{1.96\sigma}{\sqrt{n}}, \frac{S_n}{n} + \frac{1.96\sigma}{\sqrt{n}}\right]$$

with $\sigma^2 = Var(X_1)$.

The magnitude of the error is given by $\frac{1.96\sigma}{\sqrt{n}}$: the size of σ is fundamental for the speed of convergence.

Idea: Reduce σ

Find Y such that $\mathbb{E}[X_1] = \mathbb{E}[Y]$ and $Var(X_1) > Var(Y)$.

Plan

Brownian motion

We consider a probability space $(\Omega, \mathcal{A}, P)$.

Definition

Standard Brownian motion (B.M) is a stochastic process $(B_t)_{t\in[0,T]}$ fulfilling :

a) $B_0 = 0$ P-a.s.

b) B is continuous i.e $t \to B_t(w)$ is continuous for P almost all w.

c) B has independent increments: For Si $t > s$, $B_t - B_s$ is independent of $\mathcal{F}_s^{\mathcal{B}} = \sigma(B_u, u \leq s)$.

d) the increments of B are stationary and gaussian: For $t \geq s$, $B_t - B_s$ follows a $\mathcal{N}(0, t-s)$.

Brownian motion

We consider a subdivision $0 = t_0 < ... < t_n = T$ of $[0, T]$. We want to simulate

$$(\mathbf{B}_{\mathbf{t_0}}, ... \mathbf{B}_{\mathbf{t_n}}).$$

Idea:

$$\mathbf{B}_{\mathbf{t_k}} = \mathbf{B}_{\mathbf{t_{k-1}}} + \underbrace{\mathbf{B}_{\mathbf{t_k}} - \mathbf{B}_{\mathbf{t_{k-1}}}}_{\mathcal{N}(\mathbf{0}, \mathbf{t_k} - \mathbf{t_{k-1}}) \perp \mathbf{B}_{\mathbf{t_{k-1}}}, \dots, \mathbf{B_0}}.$$

Proposition

If $(G_1, ... G_n)$ *are i.i.d* $\mathcal{N}(0, 1)$, *we define*

$$X_0 = 0,\ X_i = \sum_{j=1}^{i} \sqrt{t_j - t_{j-1}} G_j \ i > 0.$$

Then

$$(X_0, ..., X_n) \underset{\mathcal{D}}{=} (B_{t_0}, ... B_{t_n}).$$

Brownian motion

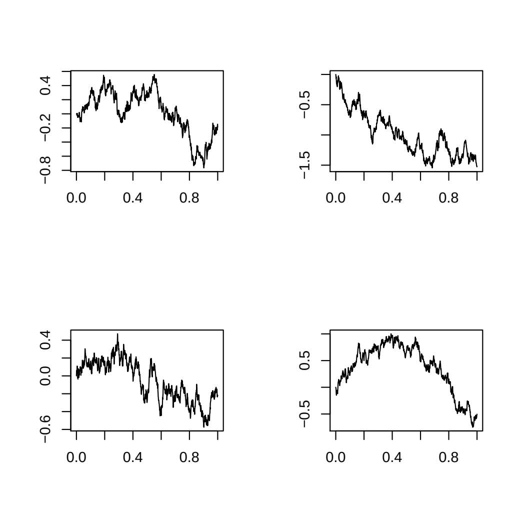

Figure: 4 paths of the Brownian motion on [0, 1] generated using the preceding method with the regular subdivision of step 0.001.

Black Scholes model

We consider the time interval $[0, T]$ and r the risk free rate (supposed to be constant) during this period.

Non-risky asset: Its dynamic is given by

$$S_0^0 = 0, \; S_t^0 = e^{rt}.$$

Risky asset: Under the historical probability P its dynamic is given by the following SDE:

$$\underline{\mathbf{dS_t = \mu S_t dt + \sigma S_t dB_t}} \qquad (1)$$

with initial condition $S_0 = x_0 > 0$ and where B is a standard BM under P.

$$\textbf{Itô formula} \quad \Rightarrow \mathbf{S_t = x_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}}.$$

Black Scholes model

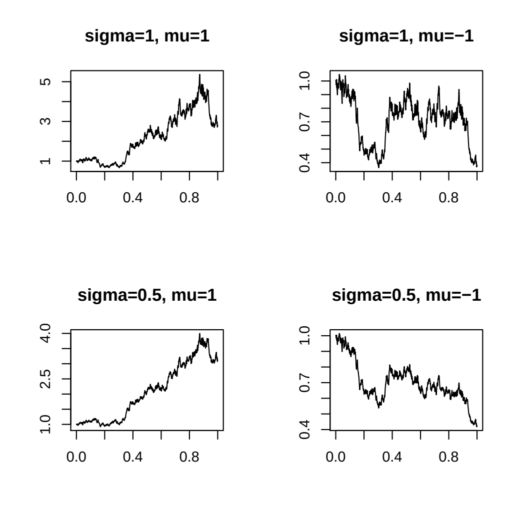

Figure: Simulation of a path of the risky asset in the B&S model for different parameters

Black Scholes model

What is, in this model, the price of a contingent claim with payoff Φ_T at T?

Proposition

In the B&S model there exists a unique probablity $Q \sim P$ such that the price at $t = 0$ of a contingent claim with payoff Φ_T at T is given by

$$price = e^{-rT}\mathbb{E}_Q[\Phi_T].$$

Moreover the dynamic of the risky asset under Q is given by

$$\underline{\mathbf{dS_t = rS_tdt + \sigma S_tdW_t}} \quad (\mu \Leftrightarrow r) \tag{2}$$

where W is a standard BM under Q.

Examples: The Black Scholes Formulas

- For Call options ($\Phi_T = (S_T - K)_+$) one has

$$\mathbb{E}_{\mathbf{Q}}[(\mathbf{S_T} - \mathbf{K})_+] = \mathbf{S_0 N(d_1)} - \mathbf{K e^{-rT} N(d_2)} \tag{3}$$

where

$$d_1 = \frac{log(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \text{ and } d_2 = \frac{log(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \tag{4}$$

and where N is the distribution function of a $\mathcal{N}(0,1)$.

- For Put options ($\Phi_T = (K - S_T)_+$) one has

$$\mathbb{E}_{\mathbf{Q}}[(\mathbf{K} - \mathbf{S_T})_+] = -\mathbf{S_0 N(-d_1)} + \mathbf{K e^{-rT} N(-d_2)}. \tag{5}$$

Black Scholes model

This formulas are fundamental because

- They are easy to compute in practice
- They are a Benchmark to test numerical methods (for example variance reduction techniques)

In the sequel, all the variance reduction methods will be test computing by Monte carlo simulations

$$\mathbb{E}[(K - e^{\sigma G})_+] \text{ or } \mathbb{E}[(e^{\sigma G} - K)_+]$$

where $G \hookrightarrow \mathcal{N}(0, 1)$ i.e computing (up to some coefficients) the price of a call

or a put option in a B&S model where $S_0 = 1$ and $r = 0$.

Black Scholes model: Numerical example

- Price of a call $\mathbb{E}[(e^{\sigma G} - K)_+]$ with $\sigma = 0.5$ and $K = 1$
  - Exact value=0.28353
  - Estimated value (N=100) = 0.249, confidence interval at 95% : [**0.164**; **0.334**]
  - Estimated value (N=1000) = 0.279, confidence interval [**0.248**; **0.308**]
  - Estimated value (N=10000) = 0.276, confidence interval [**0.267**; **0.285**]
- Price of a put $\mathbb{E}[(e^{-\sigma G} + K)_+]$ with $\sigma = 0.5$ and $K = 1$.
  - Exact value=0.15038
  - Estimated value (N=100) = 0.154, confidence interval at 95% : [**0.112**; **0.195**]
  - Estimated value (N=1000) = 0.155, confidence interval [**0.143**; **0.167**]
  - Estimated value (N=10000) = 0.149, confidence interval [**0.145**; **0.152**]

Black Scholes model

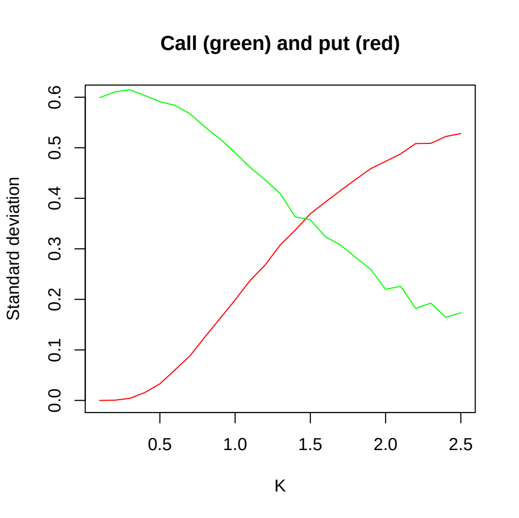

Figure: Standard deviation of the preceding payoffs ($\sigma = 0.5$) for different strikes

Black Scholes model

In practical cases, the moneyness $\frac{S_0}{K}$ of most liquid options (in our example $S_0 = 1$) belongs in general to $[0.7, 1.3]$.

Thus, It's better to price put than call by Monte Carlo methods.

To recover the price of the call we may use the Call-Put parity:

$$\mathbb{E}[(e^{\sigma G} - K)_+] = \mathbb{E}[(-e^{\sigma G} + K)_+] + e^{\frac{\sigma^2}{2}} - K.$$

- Price of a call $\mathbb{E}[(e^{\sigma G} - K)_+]$ with $\sigma = 0.5$ and $K = 1$ computed by Call-Put parity and MC
  - Exact value=0.28353
  - Estimated value (N=100) = 0.281, confidence interval at 95% : [**0.240**; **0.320**]
  - Estimated value (N=1000) = 0.285, confidence interval [**0.272**; **0.297**]
  - Estimated value (N=10000) = 0.287, confidence interval [**0.283**; **0.291**]

Plan

Controle Variate

We want to compute by Monte Carlo method $\mathbb{E}[f(X)]$.

Idea:

$$\mathbb{E}[f(X)] = \mathbb{E}[f(X) - h(X)] + \mathbb{E}[h(X)]$$

where h is chosen to ensure that

- $\mathbb{E}[h(X)]$ may be computed explicitly
- $Var(f(X) - h(X)) \ll Var(f(X))$ (intuitively we look for $f(X) - h(X)$ small).

Example: Call-Put parity

Controle Variate: Asian Options with Kemna and Vorst Method

Aim: **Price by MC method an asian call in the B&S model**

Thus we have to approximate (no closed form formula)

$$e^{-rT}\,\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_s ds - K\right)_+\right]$$

where $\forall t \in [0, T]$

$$S_t = x_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}.$$

First Idea:

$$\frac{1}{T}\int_0^T S_s ds \approx \frac{1}{p}\sum_{k=0}^{p} S_{\frac{kT}{p}} \;+\; MC$$

With $\sigma = 0.2$, $x_0 = K = 100$, $r = 0.1$, $T = 1$ and $p = 50$ one obtains

- Estimated value (N=10000) = 6.93, confidence interval at 95% : $[\mathbf{6.769}; \mathbf{7.101}]$

Controle Variate: Asian Options with Kemna and Vorst Method

Second Idea: When σ and r are small Kemna and Vorst tell us that

$$\frac{1}{T}\int_0^T S_s ds \approx exp\left(\frac{1}{T}\int_0^T log(S_s)ds\right) = A_T.$$

We are going to use the identity

$$e^{-rT}\,\mathbb{E}\left[\left(\tfrac{1}{T}\int_0^T S_s ds - K\right)_+\right] = \underbrace{e^{-rT}\,\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_s ds - K\right)_+ - (A_T - K)_+\right]}_{(1)}$$

$$+ \underbrace{e^{-rT}\,\mathbb{E}\left[(A_T - K)_+\right]}_{(2)}$$

Controle Variate: Asian Options with Kemna and Vorst Method

For (1) We use the first idea (Approximation by Riemann sums)

For (2) We just have to remark that

$$A_T = exp\left(\frac{1}{T}\int_0^T log(S_s)ds\right) \hookrightarrow exp\left(\mathcal{N}((\widetilde{r} - \frac{\widetilde{\sigma}^2}{2})T, \widetilde{\sigma}^2 T)\right)$$

where $\widetilde{r} = \frac{r}{2} - \frac{\sigma^2}{12}$ and $\widetilde{\sigma} = \frac{\sigma}{\sqrt{3}}$.

Thus we may use Black Scholes formula to compute explicitly (2).

With $\sigma = 0.2$, $x_0 = K = 100$, $r = 0.1$, $T = 1$ and $p = 50$ one obtains

- Estimated value (N=10000) = 6.981, confidence interval at 95% : $[\mathbf{6.969}; \mathbf{6.993}]$.