Stochastic Calculus: Applications in Finance

Master M2 IRFA, QEM 2

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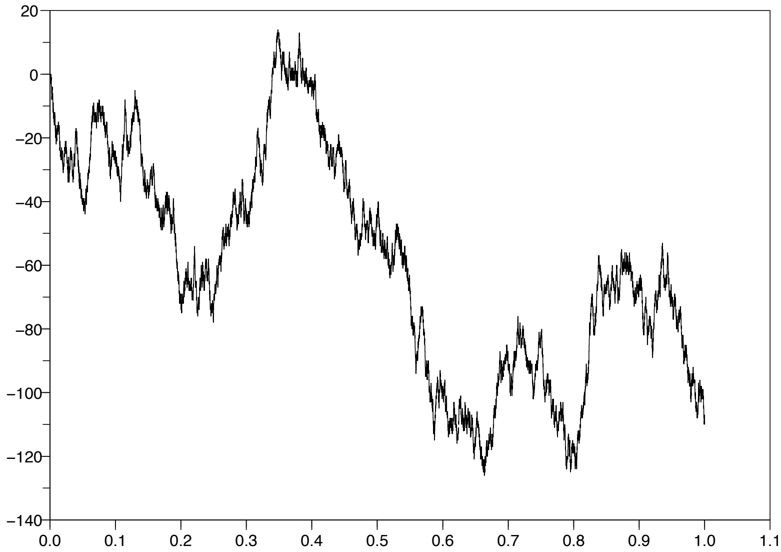
(If you see some misprints or errors please contact me at christophe.chorro@univ-paris1.fr !!!)



Contents

Reminder on Probability theory

This chapter is not an elementary introduction to Probability theory. Here we just remind some fundamental results that will be used in the sequel. For more details we refer the reader to [1] and [2].

$\forall d \in \mathbb{N}^*$, let us denote by $\mathcal{B}(\mathbb{R}^d)$ the borelian σ-algebra on $\mathbb{R}^d$, $<,>$ the usual scalar product on $\mathbb{R}^d$ and dx the Lebesgue measure on $\mathbb{R}^d$.

0.1 Introduction

0.1.1 Definition

Let $(\Omega, \mathcal{A}, P)$ be a probability space.

Definition 0.1.1 *A random variable is a function $X : (\Omega, \mathcal{A}, P) \to \mathbb{R}^d$ such that $\forall E \in \mathcal{B}(\mathbb{R}^d)$, $X^{-1}(E) \in \mathcal{A}$.*

Remark 0.1.1 *a) There exists a natural σ-algebra on Ω in order for X to be measurable. This σ-algebra is defined by $\sigma(X) = \{X^{-1}(E); E \in \mathcal{B}(\mathbb{R}^d)\}$ and is the smallest in the inclusion sense. We can show (exercise) that each random variable measurable with respect to $\sigma(X)$ has the form $h(X)$ where h is borelian.*
b) The notion of measurability is preserved by elementary algebraic operations and by passing to the limit. When Ω is a topological space equipped with its borelian σ-algebra, the continuity implies the measurability.

Definition 0.1.2 *For $A \subset \Omega$, 1_A is defined to be the function fulfilling $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \in A^c$. If $A \in \mathcal{A}$, 1_A is measurable and in this case $\sigma(A) = \{\Omega, \emptyset, A, A^c\}$.*

Random variables allow to transport probabilistic structures:

Proposition 0.1.1 *If $X : (\Omega, \mathcal{A}, P) \to \mathbb{R}^d$ is a random variable, the mapping $P_X : \mathcal{B}(\mathbb{R}^d) \to \mathbb{R}_+$ defined, $\forall E \in \mathcal{B}(\mathbb{R}^d)$, by $P_X(E) = P(X^{-1}(E))$ is a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ called the distribution of X.*

Definition 0.1.3 *We say that a random variable X has a moment of order $p \in \mathbb{N}^*$ if the quantity*

$$E[|X|^p] = \int_{\mathbb{R}^d} |x|^p dP_X(x)$$

is finite. For $p \in \mathbb{N}^*$, we define $L^p = \{X; E[|X|^p] < \infty\}$ and if $X \in L^p$, $\| X \|_p = E[|X|^p]^{\frac{1}{p}}$. Remember that $(L^p, \| X \|_p)$ is complete (because every sum that absolutely converges is convergent...).

Example 0.1.1 *When $d = 1$, we say that X follows a gaussian distribution of mean m and variance σ^2 (the standard notation is $\mathcal{N}(m, \sigma^2)$) if P_X is absolutely continuous with respect to dx and if*

$$dP_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx.$$

In this case, X has moments of all orders, in particular, $E[X] = m$ and $Var(X) = E[(X - E[X])^2] = \sigma^2$.

Exercise 0.1.1 *Let X be a random variable following a $\mathcal{N}(0, \sigma^2)$. Show that $\forall k \in \mathbb{N}$,*

$$E[X^{2k}] = \frac{(2k)!}{2^k k!} \sigma^{2k}.$$

An useful characterization of probability distributions is given by the following theorem:

Theorem 0.1.1 *a) The characteristic function $\Phi_X : t \in \mathbb{R}^d \to E[e^{i<t,X>}] \in \mathbb{C}$ characterize the distribution of X.*
b) If $d = 1$, the distribution of X is characterized by the distribution function $F_X : t \in \mathbb{R} \to P(X \leq t) \in \mathbb{R}_+$.

Example 0.1.2 *If X follows a $\mathcal{N}(m, \sigma^2)$, then*

$$\Phi_X(t) = e^{itm} e^{-\frac{\sigma^2 t^2}{2}}.$$

0.2 Notion of independence

0.2.1 Events

Definition 0.2.1 *Two events A and B (in $\mathcal{A}$) are independent if $P(A \cap B) = P(A)P(B)$. In this case, we use the classical notation $A \perp B$ (This notation will remain valid for σ-algebras and random variables).*

Definition 0.2.2 *A finite collection of events* $(A_i)_{1\leq i\leq n}$ *is an independent collection if* $P(\cap_{i\in I} A_i) = \prod_{i\in I} P(A_i)$, $\forall I \subset \{1, ..., n\}$. *Often, the events are said to be mutually independent.*

Remark 0.2.1 *Warning: If events are pairwise independent they are not mutually independent in general. In fact, considering the toss of two fair coins we can show that the events*
A="head on first coin"
B="tail on second coin"
C="The same on the two coins"
are pairwise but not mutually independent.

0.2.2 σ-algebras

Definition 0.2.3 *Sub* σ*-algebras* $(\mathcal{A}_i)_{1\leq i\leq n}$ *of* $\mathcal{A}$ *are independent if one has* $P(\cap_{1\leq i\leq n} A_i) = \prod_{1\leq i\leq n} P(A_i)$, $\forall A_i \in \mathcal{A}_i$.

0.2.3 Random variables

Definition 0.2.4 *Random variables* $(X_i)_{1\leq i\leq n}$ *are independent if the generated* σ*-algebras* $(\sigma(X_i))_{1\leq i\leq n}$ *are independent .*

We consider for notational simplicity that $d = 1$. However the results extend without difficulty to the general case.

Theorem 0.2.1 *: In order for* X_1 *and* X_2 *to be independent, it is necessary and sufficient to have any one of the following conditions holding*
a) $P_{(X_1,X_2)} = P_{X_1} \otimes P_{X_2}$
b) $\forall f, g \in C_b(\mathbb{R}, \mathbb{R})$, $E[f(X_1)g(X_2)] = E[f(X_1)]E[g(X_2)]$
c) $\Phi_{(X_1,X_2)} = \Phi_{X_1}\Phi_{X_2}$

Proposition 0.2.1 *If* X_1 *et* X_2 *are independent one has* $Var(X_1+X_2) = Var(X_1) + Var(X_2)$ *and* $E[X_1X_2] = E[X_1]E[X_2]$. *The last equality implies that two independent random variables are uncorrelated, unfortunately the converse is false in general (take* $X_1 = X$, $X_2 = X^2$ *with* X *symmetric). For more details, the reader can read the forthcoming part on gaussian vectors.*

Proposition 0.2.2 *If* X_1 *et* X_2 *are two random variables so that the pair* (X_1, X_2) *owns a density, then,* X_1 *and* X_2 *are independent if and only if the density of the pair is equal to the product of the densities of each component.*

Exercise 0.2.1 *(Box-Muller Method) Consider that U_1 and U_2 are two independent uniform random variables on the interval $[0,1]$. Show that*

$$G_1 = \sqrt{-2log(U_1)}cos(2\pi U_2) \text{ and } G_2 = \sqrt{-2log(U_1)}sin(2\pi U_2)$$

are two independent $\mathcal{N}(0,1)$.

Proof: Let us write

$$\Phi : (x,y) \in]0,1[^2 \to (u = \sqrt{-2log(x)}cos(2\pi y), v = \sqrt{-2log(x)}sin(2\pi y)) \in \mathbb{R}^2 - (\mathbb{R}_+ \times \{0\}).$$

It is easy to prove that Φ is a C^1-diffeomorphism with a Jacobian determinant fulfilling $|J(\Phi)(x,y)| = \frac{2\pi}{x}$. Since $u^2 + v^2 = -2log(x)$ thus $|J(\Phi^{-1})(u,v)| = \frac{1}{2\pi}e^{-\frac{u^2+v^2}{2}}$. According to the change of variables theorem, one has for $F \in C_b(\mathbb{R}^2, \mathbb{R})$,

$$\int_{]0,1[^2} F(\Phi(x,y))dxdy = \int_{\mathbb{R}^2-(\mathbb{R}_+\times\{0\})} F(u,v)\frac{1}{2\pi}e^{-\frac{u^2+v^2}{2}}dudv. \square$$

0.2.4 Convergence of random variables

Types of convergence

For simplicity we assume that $d = 1$.

Lemma 0.2.1 *(Borel-Cantelli) Let $(A_n)_{\mathbb{N}^*}$ be a sequence of events in $\mathcal{A}$.*

a) If $\sum_{n=1}^{+\infty} P(A_n) < \infty$, then

$$P(\{\omega \in \Omega; \omega \in A_n \text{ for an infinite number of } n\}) = 0.$$

b) If the A_n's are mutually independent with $\sum_{n=1}^{+\infty} P(A_n) = +\infty$, then,

$$P(\{\omega \in \Omega; \omega \in A_n \text{ for an infinite number of } n\}) = 1.$$

Definition 0.2.5 *We say that a sequence of random variables $(X_n)_{n\geq 0}$converges toward X almost surely ($X_n \underset{a.s}{\to} X$) if*

$$P(\{\omega \in \Omega; X_n(\omega) \underset{n\to\infty}{\to} X(\omega)\}) = 1.$$

Proposition 0.2.3 *Using B.C, we can show that $X_n \underset{a.s}{\to} X$ if $\forall \varepsilon > 0$,*

$$\sum_{n=1}^{+\infty} P(|X_n - X| > \varepsilon) < \infty.$$

Lemma 0.2.2 *Classical inequalities:*

a) (Tchebychev) If $X \in L^p$, $\lambda > 0$,

$$P(|X| > \lambda) \leq \frac{1}{\lambda^p} E[|X|^p].$$

b) (Holder) If $X \in L^p$, $Y \in L^q$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then (by concavity of the log* $ab \leq \frac{a^p}{p} + \frac{b^q}{q} \dots$*),*

$$E[|XY|] \leq \| X \|_p + \| Y \|_q \, .$$

c) (Minkowsky) If $X \in L^p$, $Y \in L^p$ *(by convexity of* x^p*)*

$$\| X + Y \|_p \leq \| X \|_p \| Y \|_p \, .$$

Definition 0.2.6 *If* $(X_n)_{n>0}$ *et* X *have finite moments of order* $p \in \mathbb{N}^*$, *we say that* $(X_n)_{n>0}$ *converges in* L^p *toward* X *(*$X_n \underset{L^p}{\rightarrow} X$*) if*

$$E[|X_n - X|^p] \underset{n\to\infty}{\rightarrow} 0.$$

Definition 0.2.7 *We say that a sequence of random variables* $(X_n)_{n>0}$ *converges toward* X *in probability (*$X_n \underset{p}{\rightarrow} X$*) if* $\forall \varepsilon > 0$

$$P(|X - X_n| > \varepsilon) \underset{n\to\infty}{\rightarrow} 0.$$

Definition 0.2.8 *We say that a sequence of random variables* $(X_n)_{n>0}$ *converges toward* X *in distribution (* $X_n \underset{\mathcal{D}}{\rightarrow} X$*) if* $\forall f \in C_b(\mathbb{R}, \mathbb{R})$,

$$E[f(X_n] \underset{n\to\infty}{\rightarrow} E[f(X)].$$

Proposition 0.2.4 *The following propositions are equivalent.*
a) $X_n \underset{\mathcal{D}}{\rightarrow} X$
b) F_{X_n} *converges pointwise to* F_X *for all point in the set of continuity of* F_X *(*$d = 1$*)*
c) Φ_{X_n} *converges pointwise to* Φ_X

Exercise 0.2.2 *Let* (G_n) *be a sequence of Gaussian random variables that converges toward* G *in* L^2. *Show that* G *is Gaussian.*

Relation between the different types of convergence

Proposition 0.2.5 *a)* $X_n \underset{a.s}{\rightarrow} X \Rightarrow X_n \underset{p}{\rightarrow} X$
b) $X_n \underset{L^1}{\rightarrow} X \Rightarrow X_n \underset{p}{\rightarrow} X$
c) $X_n \underset{p}{\rightarrow} X \Rightarrow X_n \underset{\mathcal{D}}{\rightarrow} X$
d) Si $q \geq p$, $X_n \underset{L^q}{\rightarrow} X \Rightarrow X_n \underset{L^p}{\rightarrow} X$

Proof a) comes from proposition 0.2.3.
b) Direct consequence of Tchebychev inequality.
c) Let $f \in C_b(\mathbb{R}, \mathbb{R})$ and $\varepsilon > 0$,

$$|E[f(X_n)] - E[f(X)]| \leq E[|f(X_n) - f(X)| 1_{|X_n - X| > \varepsilon}] + E[|f(X_n) - f(X)| 1_{|X_n - X| \leq \varepsilon}].$$

One has $E[|f(X_n) - f(X)| 1_{|X_n - X| > \varepsilon}] \leq cstP(|X_n - X| > \varepsilon) \underset{n \to \infty}{\rightarrow} 0$. Consider $g \in C_K(\mathbb{R}, \mathbb{R})$ so that $\| f - g \|_\infty \leq \varepsilon$,

$$E[|f(X_n) - f(X)| 1_{|X_n - X| \leq \varepsilon}] \leq 2\varepsilon + E[|g(X_n) - g(X)| 1_{|X_n - X| \leq \varepsilon}].$$

The function g being uniformly continuous, $\exists \eta > 0$ such that

$$|x - y| \leq \eta \Rightarrow |g(x) - g(y)| \leq \varepsilon,$$

thus,

$$E[|g(X_n) - g(X)| 1_{|X_n - X| \leq \varepsilon}] \leq \varepsilon + cstP(|X_n - X| \geq \eta)$$

with $P(|X_n - X| \geq \eta) \underset{n \to \infty}{\rightarrow} 0$.
d) Direct consequence of Holder inequality. □

Remark 0.2.2 *a) All the other implications are false in general.*
b) The a.s convergence is the only type that allows the usual algebraic stability. Thus, a good way to overcome mistakes is to come back to the definitions. (counter-example: $X_n = -X$ *where* X *is symmetric,* X_n *converges in distribution toward* X *and* $X_n - X$ *converges in distribution toward* $-2X$*...).*

The following results give partial converses to proposition 0.2.5.

Definition 0.2.9 *A family* $\{X_i; i \in I\}$ *of random variables in* L^1 *is said to be uniformly integrable (U.I) if*

$$\sup_{i \in I} E[|X_i| 1_{|X_i| > n}] \underset{n \to \infty}{\rightarrow} 0.$$

Example 0.2.1 *a) If I is finite $\{X_i; i \in I\}$ is U.I.*

b) if $\exists Y \in L^1$ such that $\forall i \in I$ $|X_i| \leq Y$, $\{X_i; i \in I\}$ is U.I.

Theorem 0.2.2 *If $X_n \underset{p}{\rightarrow} X$ and if the sequence (X_n) is U.I, then, $X \in L^1$ and $X_n \underset{L^1}{\rightarrow} X$.*

The following result is a direct consequence of proposition 0.2.3.

Theorem 0.2.3 *If $X_n \underset{p}{\rightarrow} X$, then there exists an extraction ϕ such that $X_{\phi(n)} \underset{a.s}{\rightarrow} X$.*

Theorem 0.2.4 *If $X_n \underset{\mathcal{D}}{\rightarrow} c$, where c is a constant then $X_n \underset{p}{\rightarrow} c$.*

Two important results

Theorem 0.2.5 *(**SLLN**) Let (X_n) be a sequence of i.i.d random variables.*
a) Suppose that $X \in L^1$. Denoting $S_n = X_1 + ... + X_n$, one has

$$\frac{S_n}{n} \underset{a.s \text{ and } L^1}{\longrightarrow} E[X_1].$$

b) If $E[|X_1|] = +\infty$ the sequence S_n diverges a.s.

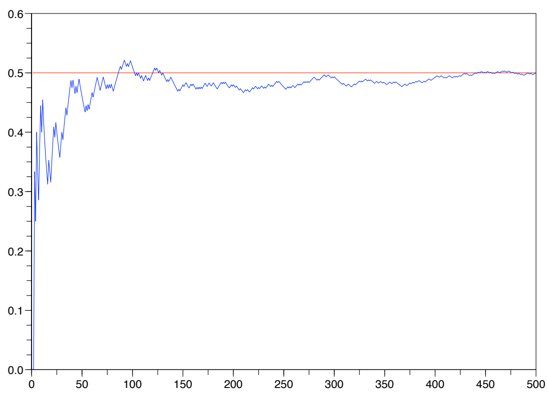
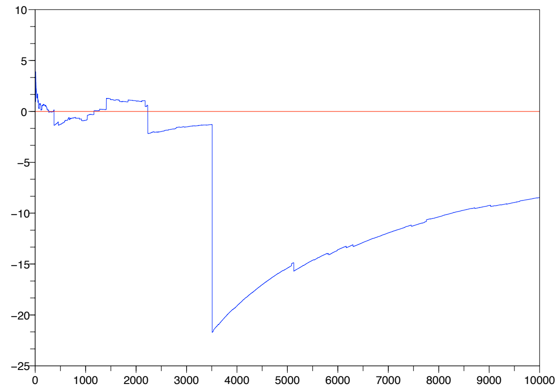


Illustration of the SLLN when $X_1 \hookrightarrow \mathcal{B}(\frac{1}{2})$ and $n = 500$



SLLN is not fulfilled when $X_1 \hookrightarrow \mathcal{C}(1)$ (here $n = 10000$)

The central limit theorem gives some precisions concerning the speed of convergence in the strong law of large numbers:

Theorem 0.2.6 *(**CLT**) Let (X_n) be a sequence of i.i.d random variables. Suppose that X_1 has finite moment of order 2 and put $m = E[X_1]$ and $\sigma^2 = Var(X_1)$. One obtains*

$$\frac{S_{n-nm}}{\sqrt{n}\sigma} \underset{\mathcal{D}}{\rightarrow} \mathcal{N}(0,1).$$

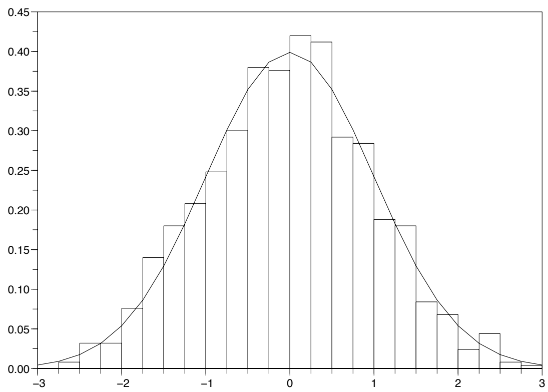


Illustration of the CLT when $X_1 \hookrightarrow \mathcal{U}([0,1])$ and $n = 500$

0.3 Gaussian vectors

Real Gaussian random variables ($\mathcal{N}(m,\sigma^2)$) has been introduced in example 0.1.1. In higher dimensions, this definition may be extended using the notion of gaussian vectors.

Definition 0.3.1 *Let $X = (X_1, ..., X_n)$ be a random vector of $\mathbb{R}^n$, X is said to be a Gaussian vector if for all $x = (x_1, ..., x_n)$ in $\mathbb{R}^n$, the real random variable $< x, X >$ is Gaussian.*

Example 0.3.1 *If $(X_1, ..., X_n)$ are independent Gaussian random variables, then, $X = (X_1, ..., X_n)$ is a Gaussian vector.*

As in dimension 1, the law of a Gaussian vector is perfectly described by two parameters:

Proposition 0.3.1 *If X is a Gaussian vector of $\mathbb{R}^n$ then, $\forall x \in \mathbb{R}^n$,*

$$\Phi_X(x) = e^{i<x,m>} e^{-\frac{x^t \Sigma x}{2}}$$

where

$$m = (E[X_1], ..., E[X_n])$$

and

$$\Sigma = [cov(X_i, X_j)]_{1 \leq i,j \leq n}.$$

In this case, we say that X follows a $\mathcal{N}(m, \Sigma)$.

Corollary 0.3.1 *(cf. prop 0.2.1) If $X = (X_1, X_2)$ is a Gaussian vector of $\mathbb{R}^2$ then*

$$X_1 \perp X_2 \Leftrightarrow cov(X_1, X_2) = 0.$$

Exercise 0.3.1 *Let $(Z, X_1, ..., X_n)$ be a Gaussian vector such that $\forall 1 \leq i \leq n$, $Z \perp X_i$. Show that $Z \perp (X_1, ..., X_n)$.*

Exercise 0.3.2 *Let $G = (G_1, ..., G_n)$ be a n-sample of a $\mathcal{N}(0,1)$, $m \in \mathbb{R}^n$ and A be a $n \times n$ matrix. Show that $m + AG$ follows a $\mathcal{N}(m, AA^t)$. Propose an algorithm to simulate an arbitrary Gaussian vector (cf. exercise 0.2.1).*

Proposition 0.3.2 *If X follows a $\mathcal{N}(m, \Sigma)$ and if Σ is nonsingular, then, X is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$ with density*

$$\frac{1}{(2\pi)^{\frac{n}{2}} (det(\Sigma))^{\frac{1}{2}}} e^{-\frac{1}{2}(x-m)^t \Sigma^{-1} (x-m)}.$$

If Σ is singular, there exist $(c_1, ..., c_n) \in \mathbb{R}^n - \{(0, ..., 0)\}$ such that

$$c_1 X_1 + ... + c_n X_n = cste \quad \text{with probability one.}$$

0.4 Conditional expectation

Let X and Y be two random variables taking values in $\mathbb{R}$. It often arises that we already know the value of X and want to calculate the expected value of Y taking into account this knowledge. This is the intuitive meaning of conditional expectation.

0.4.1 Conditioning with respect to events

Let A and B be in $\mathcal{A}$ with $P(B) > 0$. We define

$$P(A|B) =: \frac{P(A \cap B)}{P(B)}.$$

In the same way, if $X \in L^1$, we define

$$E[X|B] =: \frac{E[X1_B]}{P(B)}.$$

0.4.2 Conditioning with respect to discrete random variables

Let Y be a discrete random variable taking values $D = \{y_1, ...y_n, ...\}$ and suppose that $\forall y \in D$, $P(Y = y) > 0$.

If A belongs to $\mathcal{A}$, one defines

$$P(A|Y) =: \phi(Y) \text{ where } \ \forall y \in D,\ \phi(y) = P(A|\{Y = y\}).$$

Similarly, if $X \in L^1$, one defines

$$E[X|Y] =: \psi(Y) \text{ where } \ \forall y \in D, \psi(y) = E[X|Y = y].$$

It is very important to notice that $P(A|Y)$ and $E[X|Y]$ are random variables.

Remark 0.4.1 *: When Y is a continuous random variable, the preceding definition is meaningless because $\forall y \in \mathbb{R}$, $P(Y = y) = 0$. We have to adopt another point of view.*

0.4.3 Conditioning with respect to σ-algebras

Let $(\Omega, \mathcal{A}, P)$ be a probability space, X a random variable in L^1 and $\mathcal{G}$ a sub σ-algebra of $\mathcal{A}$.

Definition 0.4.1 *(Theorem) There exists a random variable* $Z \in L^1$ *such that*

$$i)\ Z \text{ is } \mathcal{G}\text{-measurable}$$

$$ii)\ E[XU] = E[ZU]\ \forall U\ \mathcal{G}\text{-measurable and bounded.}$$

Z is denoted by $E[X|\mathcal{G}]$ and is called the conditional expectation of X given $\mathcal{G}$. Moreover, Z is unique up to a.s equality.

Remark 0.4.2 *: The preceding result is based on the powerful Radon Nikodym theorem (paragraph 0.6). Nevertheless, when $X \in L^2$, the existence of Z can be proved using Hilbert space methods. If fact, considering the orthogonal projection Π of $L^2(\Omega, \mathcal{A}, P)$ on the closed convex set $L^2(\Omega, \mathcal{G}, P)$ we show easily that $\Pi(X) = E[X|\mathcal{G}]$. Intuitively, $E[X|\mathcal{G}]$ is the best approximation of X by $\mathcal{G}$-measurable random variables.*

0.4.4 Conditioning with respect to general random variables

Let Y be a random variable.

Definition 0.4.2 *When $X \in L^1$, $E[X|Y]$ is defined to be $E[X|\sigma(Y)]$.*

According to remark 0.1.1, $E[X|Y]$ is of the form $\psi(Y)$ where $\psi : \mathbb{R} \to \mathbb{R}$ is borelian. Naturally, this definition coincides with the one proposed in paragraph 0.4.2 in the discrete case.

Remark 0.4.3 *Using remark 0.1.1, the preceding definition may be rewritten in the following way*

$$i)\ E[X|Y] \text{ is } \sigma(Y)\text{-measurable}$$

$$ii)\ E[Xg(Y)] = E[E[X|Y]g(Y)]\ \forall g \text{ borelian and bounded.}$$

0.4.5 Conditional distribution

Let us consider a pair (X, Y) of real random variables in L^1. Suppose that this pair owns a density $f_{(X,Y)}$ with respect to the Lebesgue measure on $\mathbb{R}^2$. Thus, the marginal densities of X and Y are given by

$$f_X(x) = \int_{\mathbb{R}} f_{(X,Y)}(x, y)dy \text{ and } f_Y(y) = \int_{\mathbb{R}} f_{(X,Y)}(x, y)dx.$$

When $X \perp Y$, one has $f_{(X,Y)} = f_X f_Y$. If the condition of independence is relaxed, we obtain the following disintegration formula : $f_{(X,Y)}(x, y) = f_{X|Y}(x, y)f_Y(y)$ where

$$f_{X|Y}(x, y) = \frac{f_{(X,Y)}(x, y)}{f_Y(y)}$$

if $f_Y(y) \neq 0$ and $f_{X|Y}(x,y) = 0$ otherwise. The function $f_{X|Y}$ is called the conditional density of X given Y. In fact, if ϕ satisfies $\phi(X) \in L^1$,

$$E[\phi(X)|Y] = \Phi(Y) \text{ with } \Phi(y) = \int_{\mathbb{R}} \phi(x) f_{X|Y}(x,y)dx.$$

0.4.6 Properties of conditional expectation

Classical properties of expectation

Proposition 0.4.1 *Let X and Y be two random variables in L^1 and $\mathcal{G}$ a sub σ-algebra of $\mathcal{A}$.*

a) (positivity) If $X \geq 0$ P-a.s, $E[X|\mathcal{G}] \geq 0$ P-a.s.

b) (linearity) If $(\alpha, \beta) \in \mathbb{R}^2$, $E[\alpha X + \beta Y|\mathcal{G}] = \alpha E[X|\mathcal{G}] + \beta E[Y|\mathcal{G}]$ P-a.s.

c) (monotony) If $X \geq Y$ P-a.s, $E[X|\mathcal{G}] \geq E[Y|\mathcal{G}]$ P-a.s. (Consider $\{E[Y|\mathcal{G}] - E[X|\mathcal{G}] \geq \varepsilon\}$.)

d) (Beppo-lévy) If X_n is a sequence of non negative random variables in L^1 with $X_n \uparrow X$ P-a.s, then $E[X_n|\mathcal{G}] \uparrow E[X|\mathcal{G}]$ P-a.s.

e) (Fatou) If X_n is a sequence of non negative random variables in L^1, then $E[\liminf_n X_n|\mathcal{G}] \leq \liminf_n E[X_n|\mathcal{G}]$ P-a.s.

f) (Dominated convergence) If X_n is a sequence of random variables in L^1 such that $X_n \underset{a.s}{\to} X$ and $\forall n \in \mathbb{N}$, $X_n \leq X \in L^1$, then, $E[X_n|\mathcal{G}] \underset{a.s \text{ and } L^1}{\to} E[X|\mathcal{G}]$.

g) (Jensen) If $\psi : \mathbb{R} \to \mathbb{R}$ is a convex function such that $\psi(X) \in L^1$, then, $\psi(E[X|\mathcal{G}]) \leq E[\psi(X)|\mathcal{G}]$ P-a.s.

Remark 0.4.4 *We deduce from g) that the conditional expectation is a contracting operator on L^p spaces.*

Specific properties

Proposition 0.4.2 *Let X and Y be two random variables in L^1 and $\mathcal{G}$ a sub σ-algebra of $\mathcal{A}$.*

a) $E[E[X|\mathcal{G}]] = E[X]$

b) If X is $\mathcal{G}$-measurable, $E[X|\mathcal{G}] = X$ P-a.s.

c) (Taking out what is known) If Y is $\mathcal{G}$-measurable and bounded, $E[XY|\mathcal{G}] = YE[X|\mathcal{G}]$ P-a.s.

d) (Role of independence) If $\sigma(X)\perp\mathcal{G}$, $E[X|\mathcal{G}] = E[X]$ P-a.s.

e) (Tower property) If $\mathcal{G}'$ is a sub σ-algebra of $\mathcal{A}$ such that $\mathcal{G}' \subset \mathcal{G}$, then, $E[E[X|\mathcal{G}]|\mathcal{G}'] = E[X|\mathcal{G}']$ P-a.s.

f) If $(\mathcal{G}_i)_{i\in I}$ is a collection of sub σ-algebras of $\mathcal{A}$, the family $(E[X|\mathcal{G}_i])_{i\in I}$ is U.I.

Proof: We only give the proof of f), the others are left to the reader. According to Jensen inequality for conditional expectation,

$$E[|E[X|\mathcal{G}_i]|1_{|E[X|\mathcal{G}_i]|>n}] \leq E[E[|X|\ |\mathcal{G}_i]1_{|E[X|\mathcal{G}_i]|>n}] = E[|X|1_{|E[X|\mathcal{G}_i]|>n}].$$

But, from Tchebychev inequality one obtains

$$P(|E[X|\mathcal{G}_i]| > n) \leq \frac{E[|E[X|\mathcal{G}_i]|]}{n} \leq \frac{E[E[|X|\ |\mathcal{G}_i]]}{n} = \frac{E[|X|]}{n}.$$

We conclude using the following lemma:

Lemma 0.4.1 *Si $X \in L^1$, $\forall\varepsilon > 0$, $\exists\delta > 0$, $P(A) < \delta \Rightarrow E[|X|1_A] < \varepsilon$.*

□

The preceding properties are often sufficient to compute conditional expectations. We only come back to the definition for more difficult cases as we can see in the following proposition.

The following result will be useful in the study of the Black-Scholes model and more generally to prove Markov properties of stochastic processes.

Proposition 0.4.3 *If $\sigma(X)\perp\mathcal{G}$ and if Y is $\mathcal{G}$-measurable, then, for all borelian function $\Phi : \mathbb{R}^2 \to \mathbb{R}$ such that $E[|\Phi(X,Y)|] < \infty$, one has*

$$E[\Phi(X,Y)|\mathcal{G}] = \psi(Y) \text{ where } \psi(y) = E[\Phi(X,y)].$$

Proof: We have $\psi(y) = \int_{\mathbb{R}} \Phi(x,y)dP_X(x)$ and the measurability of ψ is a classical consequence of Fubini's theorem. For $G \in \mathcal{G}$, we set $Z = 1_G$. We deduce from the hypotheses that $P_{(X,Y,Z)} = P_X \otimes P_{(Y,Z)}$, thus,

$$E[\Phi(X,Y)1_G] = \int_{\mathbb{R}}\int_{\mathbb{R}^2} \Phi(x,y)z dP_{(Y,Z)}(y,z)dP_X(x).$$

By Fubini's theorem,

$$E[\Phi(X,Y)1_G] = \int_{\mathbb{R}^2} \psi(y) z dP_{(Y,Z)}(y,z)$$

so

$$E[\Phi(X,Y)1_G] = E[\psi(Y)1_G].$$

□

0.4.7 Conditional expectation and Gaussian vectors

Proposition 0.4.4 *Let $(Z, X_1, ..., X_n)$ be a Gaussian vector. Then, there exist real numbers $(a, b_1, ..., b_n)$ such that*

$$E[Z|X_1, ..., X_n] = a + \sum_{i=1}^{n} X_i b_i.$$

Proof: Consider the closed sub vector space (of finite dimension) of L^2 spanned by $(1, X_1, ..., X_n)$. Let Π denotes the orthogonal projection on this closed and convex set. Thus, there exist real numbers $(a, b_1, ..., b_n)$ such that $\Pi(Z) = a + \sum_{i=1}^{n} X_i b_i$. For $Y = Z - \Pi(Z)$, one obtains classically $E[Y] = 0$ and $E[YX_i] = 0$. In this way, $cov(Y, X_i) = 0$. The random vector (Y, X_i) being Gaussian, we deduce from corollary 0.3.1, $Y \perp X_i$. By exercise 0.3.1, $Y \perp (X_1, ..., X_n)$ and $E[Y|X_1, ..., X_n] = E[Y] = 0$, thus, $E[Z|X_1, ..., X_n] = \Pi(Z) = a + \sum_{i=1}^{n} X_i b_i$.□

0.5 Stochastic processes

0.5.1 Introduction

In order to model systems depending on time and hazard the natural mathematical object are stochastic processes: a probability space $(\Omega, \mathcal{A}, P)$ and a function $(t, \omega) \to X_t(\omega)$. For fixed t, the state of the system is a random variable X_t, on the other hand, a particular evolution of this system (i.e for fixed ω) is represented by the function $t \to X_t(\omega)$ called a trajectory (or a sample path).

Definition 0.5.1 *A stochastic process on $(\Omega, \mathcal{A}, P)$, indexed by an arbitrary set $T \subset \mathbb{R}_+$, is a collection $(X_t)_{t \in T}$ of random variables on $(\Omega, \mathcal{A}, P)$ with values on a space E (for us, $E = \mathbb{R}$).*

Several notions exist to compare stochastic processes taking into account time evolution.

Definition 0.5.2 *Two stochastic processes $(X_t)_{t\in T}$ and $(Y_t)_{t\in T}$ are equidistributed if, $\forall n \in \mathbb{N}^*$, $\forall (t_1, ..., t_n) \in T^n$*

$$(X_{t_1}, ..., X_{t_n}) \underset{\mathcal{D}}{=} (Y_{t_1}, ..., Y_{t_n}).$$

In this case we also say that they have the same finite-dimensional distributions.

Definition 0.5.3 *A stochastic process $(X_t)_{t\in T}$ is a version of the process $(Y_t)_{t\in T}$ if, $\forall t \in T$,*

$$X_t = Y_t \ P - a.s.$$

Such processes are also said to be stochastically equivalent.

Definition 0.5.4 *Two stochastic processes $(X_t)_{t\in T}$ and $(Y_t)_{t\in T}$ are indistinguishable (or equivalent up to evanescence) when*

$$P(\{\omega | \forall t \in T, \ X_t(\omega) = Y_t(\omega)\}) = 1$$

Remark 0.5.1 *These definitions are more and more restrictive. (exercises)*

Definition 0.5.5 *We say that the sample paths of a stochastic process are continuous (or monotonic, or cadlag) if, for P almost all ω, $t \in T \to X_t(\omega)$ is continuous (or monotonic, or cadlag). For simplicity we say that the process is continuous (or monotonic, or cadlag).*

Exercise 0.5.1 *Let $(X_t)_{t\in T}$ and $(Y_t)_{t\in T}$ be stochastic processes.*

1) Supposing that $(X_t)_{t\in T}$ and $(Y_t)_{t\in T}$ are right continuous, show that if $(X_t)_{t\in T}$ is a version of $(Y_t)_{t\in T}$, then they are indistinguishable.

2) On the probability space $([0,1], \mathcal{B}([0,1]), dx)$, let $(X_t)_{t\in\mathbb{R}_+}$ be the process defined by $X_t(\omega) = \omega + at$ and $(Y_t)_{t\in\mathbb{R}_+}$ the process defined by $Y_t(\omega) = \omega + at$ if $t \neq \omega$ and $Y_t(\omega) = 0$ otherwise. Show that $(X_t)_{t\in\mathbb{R}_+}$ is a version of $(Y_t)_{t\in\mathbb{R}_+}$ but that they are not indistinguishable.

For simplicity, we suppose from now on that $T = \mathbb{R}_+$. Let $(X_t)_{t\in T}$ be a stochastic process. For fixed times $0 \leq t_1 \leq ... \leq t_n$ we denote by $P_{t_1,...,t_n}$ the distribution of the random vector $(X_{t_1}, ..., X_{t_n})$. Remark that for all collection of borelian sets $(A_1, ..., A_n)$ and for $t_{n+1} \geq t_n$, one has

$$P_{t_1,...,t_n}(A_1 \times ... \times A_n) = P_{t_1,...,t_n,t_{n+1}}(A_1 \times ... \times A_n \times \Omega). \quad (1)$$

The following result due to Kolmogorov ensures the existence of a stochastic process related to a family of finite-dimensional marginal distributions provided that a natural consistency condition of type (1) is fulfilled. This theorem is very useful to show the existence of particular stochastic processes.

Theorem 0.5.1 *Consider a collection of distributions*

$$\{P_{t_1,...,t_n}; n \geq 1, 0 \leq t_1 \leq ... \leq t_n\}$$

such that:

a) $P_{t_1,...,t_n}$ *is a distribution on* $\mathbb{R}^n$

b) If $\{0 \leq s_1 \leq ... \leq s_m\} \subset \{0 \leq t_1 \leq ... \leq t_n\}$ *then*

$$\pi_* P_{t_1,...,t_n} = P_{s_1,...,s_m}$$

where π *is the natural projection from* $\mathbb{R}^n$ *on* $\mathbb{R}^m$.

There exists a stochastic process $(X_t)_{t \in \mathbb{R}_+}$ *with marginal finite-dimensional distributions given by the* $\{P_{t_1,...,t_n}\}'s$.

A stochastic process being a random function depending on two arguments, we have the following notion of measurability.

Definition 0.5.6 *A stochastic process* $(X_t)_{t \in \mathbb{R}_+}$ *is measurable if the function*

$$(t, \omega) \in (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)) \times (\Omega, \mathcal{A}) \to X_t(\omega) \in (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

is measurable i.e $\forall T \in \mathbb{R}_+$, $\forall E \in \mathcal{B}(\mathbb{R})$,

$$\{(t, \omega); 0 \leq t \leq T, X_t(\omega) \in E\} \in \mathcal{B}([0, T]) \otimes \mathcal{A}.$$

Remark 0.5.2 *If* $(X_t)_{t \in \mathbb{R}_+}$ *is measurable and* $f \in C_b(\mathbb{R}, \mathbb{R})$, *the random variable* $Y_t = \int_0^t f(X_s) ds$ *is well defined.*

Exercise 0.5.2 *Show that a right continuous process (or a left continuous one) is measurable. (Hint: if* $t \leq T$, *remark that* $X_t = X_0 + lim_n X_n$ *with* $X_n = \sum_{k=0}^{2^n-1} X_{\frac{(k+1)T}{2^n}} 1_{]\frac{kT}{2^n}, \frac{(k+1)T}{2^n}]}$.*)*

Definition 0.5.7 *Classically we denote*

$$L^2(\Omega \times [0, T]) = \left\{ (X_t)_{t \in [0,T]} \text{ measurable}; E\left[\int_0^T X_s^2 ds\right] < \infty \right\}$$

the Hilbert space of squared integrable stochastic processes.

0.5.2 Filtrations, adapted processes

Let $(\Omega, \mathcal{A}, P)$ be a probability space.

Definition 0.5.8 *A non decreasing collection $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ of sub σ-algebras of $\mathcal{A}$ is called a filtration. This filtration is said to be complete if $\forall t \in \mathbb{R}_+$, $\mathcal{N} \subset \mathcal{F}_t$ where*

$$\mathcal{N} = \{N \subset \Omega; \exists A \in \mathcal{A}, N \subset A, P(A) = 0\}.$$

Remark 0.5.3 *Intuitively, $\mathcal{F}_t$ represents the information available at time t. Moreover, we may always boil down to the complete case changing $\mathcal{F}_t$ into $\sigma(\mathcal{N} \cup \mathcal{F}_t)$.*

From now on, according to the preceding remark, we will suppose that all the considered filtrations are complete.

Definition 0.5.9 *A stochastic process $(X_t)_{t\in\mathbb{R}_+}$ is adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ if X_t is $\mathcal{F}_t$ measurable.*

Remark 0.5.4 *A stochastic process $(X_t)_{t\in\mathbb{R}_+}$ is always adapted to its natural filtration $\mathcal{F}_t^X = \sigma(X_s; 0 \leq s \leq t)$.*

Remark 0.5.5 *There are several advantages to work with complete filtrations, in particular,*

a) if $X_t \underset{a.s}{=} Y_t$ and if X_t is $\mathcal{F}_t$ measurable $\Rightarrow$ Y_t is $\mathcal{F}_t$ measurable (thus, any version of an adapted process is an adapted process).

b) if $X_t^n \underset{a.s}{\rightarrow} X_t$ and if $\forall n \geq 0$, X_t^n is $\mathcal{F}_t$ measurable $\Rightarrow$ then X_t is $\mathcal{F}_t$ measurable.

The preceding dynamic (related to a filtration) notion of measurability is restrictive. In fact, it omits to take into account that a stochastic process is a random function depending on two arguments.

Definition 0.5.10 *A stochastic process $(X_t)_{t\in\mathbb{R}_+}$ is progressively measurable if, $\forall T > 0$ the function*

$$(t, \omega) \in ([0, T], \mathcal{B}([0, T])) \times (\Omega, \mathcal{F}_T) \rightarrow X_t(\omega) \in (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

is measurable.

Exercise 0.5.3 *For $0 \leq t_1 \leq ... \leq t_n = T$, let F_{t_i} be a $\mathcal{F}_{t_i}$-measurable random variable and*

$$X_t = \sum_{i=1}^{n-1} F_{t_i} 1_{[t_i, t_{i+1}[(t)}. \tag{2}$$

Show that $(X_t)_{t\in[0,T]}$ is progressively measurable. We will denote $\mathcal{E}([0, T] \times \Omega)$ for the elements of the form (2) with $F_{t_i} \in L^2(\mathcal{F}_{t_i})$.

Remark 0.5.6 *A progressively measurable process is measurable and adapted.*

Exercise 0.5.4 *When $(X_t)_{t\in\mathbb{R}_+}$ is progressively measurable show that the process $(Y_t)_{t\in\mathbb{R}_+}$ defined in remark 0.5.2 is adapted (Use Fubini's theorem).*

Proposition 0.5.1 *A right continuous process $(X_t)_{t\in\mathbb{R}_+}$ (or left continuous) is progressively measurable when it is adapted.*

Proof: In the left continuous case we put $X_n(t) = X_{[n\frac{t}{T}]\frac{T}{n}}$. One has $\forall t \in \mathbb{R}_+$, $X_t^n \underset{a.s}{\rightarrow} X_t$. Now, $\forall B \in \mathcal{B}(\mathbb{R})$,

$$\{(t,\omega); 0 \le t \le T, X_n(t) \in B\} = [0, \frac{T}{n}[\times\{X_0 \in B\} \cup ..... \in \mathcal{B}([0,T]) \times \mathcal{F}_T$$

and the result follows. In the right continuous case we use the same approach with $X_n(t) = X_{inf(T,([n\frac{t}{T}]+1)\frac{T}{n})}$. $\square$

Definition 0.5.11 *We denote*

$$L^2_{prog}(\Omega\times[0,T]) = \left\{(X_t)_{t\in[0,T]} \text{ prog meas}; E\left[\int_0^T X_s^2 ds\right] < \infty\right\}.$$

Theorem 0.5.2 *The space $L^2_{prog}(\Omega\times[0,T])$ equipped with its natural norm is complete. Moreover, $\mathcal{E}([0,T]\times\Omega)$ is dense in $L^2_{prog}(\Omega\times[0,T])$.*

Proof: We only prove the second part of the theorem, the first one being a classical result of probability theory.

First, let us introduce an approximation procedure in the deterministic case. When $f \in L^2([0,T],dx)$, we define $\forall n \in \mathbb{N}^*$, $\forall t \in [0,T]$,

$$P_n(f)(t) = n\sum_{i=1}^{n-1}\int_{\frac{i-1}{n}}^{\frac{i}{n}} f(s)ds 1_{]\frac{i}{n},\frac{i+1}{n}]}(t).$$

This linear operator P_n is a contraction, in fact, if $t \in]\frac{i}{n},\frac{i+1}{n}]$

$$[P_n(f)(t)]^2 \le n\int_{\frac{i-1}{n}}^{\frac{i}{n}} f^2(s)ds$$

thus

$$\int_0^T [P_n(f)(t)]^2 dt \le \int_0^T [f(t)]^2 dt.$$

We can also show that $\forall f \in C_K([0,T],\mathbb{R})$,

$$P_n(f) \underset{L^2([0,T])}{\rightarrow} f \quad (3)$$

and extend this result for $f \in L^2([0,T])$ by density.

Now this method is extended to $L^2_{prog}(\Omega \times [0,T])$. We put $\forall X \in L^2_{prog}(\Omega \times [0,T])$, $\forall t \in [0,T]$, for almost all ω,

$$P_n(X_t(w)) = P_n(X_.(\omega))(t).$$

Since $X \in L^2_{prog}(\Omega \times [0,T])$, $P_n(X) \in \mathcal{E}([0,T] \times \Omega)$ because (exercise 0.5.4) $\int_{\frac{i-1}{n}}^{\frac{i}{n}} X_s ds$ is $\mathcal{F}_{\frac{i}{n}}$ measurable. Now it is easy to prove, using (3) and the dominated convergence theorem, the convergence of $P_n(X)$ toward X in $L^2_{prog}(\Omega \times [0,T])$: X belonging to $L^2_{prog}(\Omega \times [0,T])$, for almost all ω, $X_.(\omega) \in L^2([0,T])$, thus, we deduce from (3) that

$$\int_0^T (P_n(X_s) - X_s)^2 ds \underset{a.s}{\rightarrow} 0$$

being bounded (Minkowski inequality) by $4\|X_.(\omega)\|^2_{L^2([0,T])} \in L^2(P)$. □

0.5.3 Gaussian processes

Definition 0.5.12 *A stochastic process $(X_t)_{t\in\mathbb{R}_+}$ is Gaussian if, $\forall n \in \mathbb{N}^*$, $\forall t_1, ..., t_n \in \mathbb{R}_+$, the random vector $(X_{t_1}, ..., X_{t_n})$ is Gaussian.*

The distribution of a gaussian process is entirely described by two functional parameters, the mean function $m : t \rightarrow E[X_t]$ and the covariance function $\Gamma : (s,t) \rightarrow E[(X_t - m(t))(X_s - m(s))]$ that is symmetric and non-negative in the sense that $\forall n \in \mathbb{N}^*$, $\forall t_1, ..., t_n \in \mathbb{R}_+$, the square matrix $[\Gamma(t_i,t_j)]_{1\leq i,j\leq n}$ is symmetric non-negative.

In fact we have the following result derived from the Kolmogorov theorem.

Proposition 0.5.2 *Let $m : \mathbb{R}_+ \rightarrow \mathbb{R}$ be an arbitrary function and $\Gamma : (\mathbb{R}_+)^2 \rightarrow \mathbb{R}$ a symmetric and non-negative function. Then, there exists a Gaussian process with mean function m and covariance function Γ.*

Example 0.5.1 *Show that $(s,t) \in (\mathbb{R}_+)^2 \rightarrow inf(s,t)$ is non-negative. (Hint: You may use an induction reasoning)*

0.5.4 Martingales in continuous time

We refer the reader to [6] for an overview of the theory of martingales.

Let $(\Omega, \mathcal{A}, P)$ a probability space and $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ a filtration on $\mathcal{A}$.

Definition 0.5.13 *An adapted process $(M_t)_{t\in\mathbb{R}_+}$ with values in L^1 is:*

- *a martingale if $\forall t \geq s$, $E[M_t|\mathcal{F}_s] = M_s$.*

- *a supermartingale if, $\forall t \geq s$, $E[M_t|\mathcal{F}_s] \leq M_s$.*

- *a submartingale if, $\forall t \geq s$, $E[M_t|\mathcal{F}_s] \geq M_s$.*

Remark 0.5.7 *A martingale (M_t) fulfills $\forall t \geq 0$ $E[M_t] = E[M_0]$.*

Example 0.5.2 *If $X \in L^1$, $M_t =: E[X|\mathcal{F}_t]$ is, according to proposition 0.4.2, a martingale.*

The following results will be useful in the sequel.

Proposition 0.5.3 *Let $(M_t)_{t\in\mathbb{R}_+}$ be a square integrable martingale, then, $\forall s \leq t$, one has*

$$E[(M_t - M_s)^2|\mathcal{F}_s] = E[M_t^2 - M_s^2|\mathcal{F}_s].$$

Proof: One has

$$E[(M_t - M_s)^2|\mathcal{F}_s] = E[M_t^2|\mathcal{F}_s] - 2E[M_tM_s|\mathcal{F}_s] + \underbrace{E[M_s^2|\mathcal{F}_s]}_{M_s^2}.$$

The result follows from $E[M_tM_s|\mathcal{F}_s] = M_s^2$.□

Proposition 0.5.4 *A square integrable martingale $(M_t)_{t\in\mathbb{R}_+}$ has orthogonal increments.*

Proof: One has to prove that for $t_4 > t_3 \geq t_2 > t_1$,

$$E[(M_{t_4} - M_{t_3})(M_{t_2} - M_{t_1})] = 0.$$

This equality is obtained conditioning by $\mathcal{F}_{t_3}$.□

Proposition 0.5.5 *Let $(M_t)_{t\in\mathbb{R}_+}$ be a square integrable martingale such that there exists $(\Phi_t)_{t\in[0,T]} \in L^2(\Omega \times [0,T])$ fulfilling $\forall 0 \leq t \leq T$, $M_t = \int_0^t \Phi_s ds$. Then*

$$P(\{\omega; \forall 0 \leq t \leq T, M_t = 0\}) = 1.$$

Proof: According to the preceding proposition,

$$E[M_t^2] = E\left[\left(\sum_{i=1}^{n} M_{\frac{it}{n}} - M_{\frac{(i-1)t}{n}}\right)^2\right] = E\left[\sum_{i=1}^{n}\left(M_{\frac{it}{n}} - M_{\frac{(i-1)t}{n}}\right)^2\right].$$

From Schwartz's inequality,

$$E[M_t^2] = E\left[\sum_{i=1}^{n}\left(\int_{\frac{(i-1)t}{n}}^{\frac{it}{n}} \Phi_s ds\right)^2\right] \leq \frac{t}{n} E\left[\int_0^t \Phi_s^2 ds\right].$$

Let n go to infinity and use the continuity of the process $(M_t)_{t\in\mathbb{R}_+}$ give the result.□

For simplicity, the notion of stopping time is not tackled in this lecture even if it is fruitful in martingale theory. However we state the following fundamental result.

Theorem 0.5.3 *(Doob's inequality) Let $(M_t)_{t\in\mathbb{R}_+}$ be a square integrable martingale with continuous paths, then, $\forall T \in \mathbb{R}_+$,*

$$E[\sup_{0\leq t\leq T} |M_t|^2] \leq 4E[|M_T|^2].$$

From now on, $M^2([0,T])$ denotes the space of square integrable martingales on $[0,T]$ with continuous trajectories (quotiented by the equivalence relation $M \sim M'$ if and only if M and M' are indistinguishable). If $(M_t)_{t\in[0,T]} \in M^2([0,T])$ we denote by $\|\|_{M^2}$ the function $\|M\|_{M^2} = E[|M_T|^2]^{\frac{1}{2}}$.

Proposition 0.5.6 *The function $\|\|_{M^2}$ defined on $M^2([0,T])$ is a norm. The space $M^2([0,T])$ equipped with $\|\|_{M^2}$ is an Hilbert space.*

Proof: We admit the second point. The first one is a direct consequence of Doob's inequality.□

Remark 0.5.8 *For the completeness of $M^2([0,T])$, the completeness of the filtration is necessary.*

0.6 Radon-Nikodym's theorem

Definition 0.6.1 *Let P and Q be two probability measures defined on the same probability space $(\Omega, \mathcal{A})$.*

a) P is said to be absolutely continuous with respect to Q (notation: $P \ll Q$) if $\forall A \in \mathcal{A}$, $Q(A) = 0 \Rightarrow P(A) = 0$.
b) P and Q are said to be equivalent (notation: $P \sim Q$) if $\forall A \in \mathcal{A}$, $P(A) = 0 \Leftrightarrow Q(A) = 0$.
c) P and Q are said to be singular (notation: $P \perp Q$) if $\exists A \in \mathcal{A}$, such that $P(A) = 0$ and $Q(A) = 1$.

The following result will be useful for a good understanding of change of probabilities in financial models.

Theorem 0.6.1 *(Radon-Nikodym) Let P and Q be two probability measures defined on the same probability space $(\Omega, \mathcal{A})$. Then $P \ll Q$ if and only if there exists a random variable $Z \geq 0$ $Q-$integrable (unique up to Q a.s equality) fulfilling $E_Q[Z] = 1$ such that, $\forall A \in \mathcal{A}$,*

$$P(A) = E_Q[Z 1_A].$$

The random variable Z is called the Radon-Nikodym derivative of P with respect to Q. Usually it is denoted by $Z =: \frac{dP}{dQ}$.

Remark 0.6.1 *If $P \sim Q$ one has $Z > 0$ Q (or P) a.s, in this case, $\frac{dP}{dQ} = \frac{1}{\frac{dQ}{dP}}$.*

Exercise 0.6.1 *The aim is to prove the theorem-definition 0.4.1. For $X \in L^1(\Omega, \mathcal{A}, P)$, we want to show the existence of $E[X|\mathcal{G}]$.*

a) Prove that we can suppose $X \geq 0$.

b) Show that the function defined on $(\Omega, \mathcal{A})$ by

$$Q(A) = \int_A X dP$$

is a bounded non-negative measure such that $Q \ll P$.

c) Show that the same holds if Q is restricted to $(\Omega, \mathcal{G})$.

d) Using Radon-Nikodym theorem, propose a natural candidate for $E[X|\mathcal{G}]$.

Example 0.6.1 *(A first step toward Girsanov theorem) Consider a random variable X following, under a probability P, a $\mathcal{N}(m, \sigma^2)$. Here we want to find a probability Q equivalent to P such that, under Q, X follows a $\mathcal{N}(0, \sigma^2)$. Consider the random variable*

$$L = e^{-\frac{mX}{\sigma^2}} e^{+\frac{m^2}{2\sigma^2}}.$$

According to example 0.1.2, we can see easily that $E_P[Z] = 1$*, thus we define the probability* Q *by* $L = \frac{dQ}{dP}$*. Since*

$$E_Q[e^{itX}] = E_P[Le^{itX}] = e^{-\frac{\sigma^2 t^2}{2}}$$

the result follows.

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Chapter 1

Brownian motion

1.1 Short history

We refer the reader to [9] for an historical overview.

Brownian motion is probability the most famous and the most important stochastic process. It is a beautiful example of fruitful links between mathematics and physics.

Brownian motion is generally regarded as having been discovered by the botanist Robert Brown in 1827. While Brown was studying pollen particles floating in water in the microscope, he observed tiny particles in the pollen grains executing a jittery motion. After repeating the experiment with particles of dust, he was able to conclude that the motion wasn't due to pollen being "alive" but the origin of the motion remained unexplained. Later (1877) this phenomenon was partially explained by Delsaux: Due to the thermal agitation, a small pollen particle would receive a random number of impacts of random strength and from random directions in any short period of time. This random bombardment by the big molecules of the fluid would cause a sufficiently small particle to move exactly just how Brown described it. (See "http://chaos.nus.edu.sg/simulations/" for an interesting numerical simulation). The first one to give a theory of Brownian motion was Louis Bachelier in 1900 in his PhD thesis "The theory of speculation" (see [2]). He used this object for modeling assets on which contracts trade and underline its Markov properties.

However, it was only in 1905 ([6]) that Albert Einstein, using a probabilistic model, proposed a "good" mathematical definition and presented it as a way to indirectly confirm the existence of atoms and molecules:

Let X_t be the position at time t of a small particle in a fluid. Suppose that

a) $X_{t+h} - X_t$ is independent of $\sigma(X_s; s \leq t)$.

b) the distribution of $X_{t+h} - X_t$ doesn't depend on t.

c) the trajectories are continuous.

The first hypothesis means that the evolution of the trajectory during the interval $[t, t+h]$ is only due to the thermal impacts during this period. The inertia (i.e the mass of the particle) is neglected.

The second one implies that the physical environment is the same along the time (no variations of temperature).

The third one ensures that the particle doesn't jump.

We have to remark for the moment that no Gaussian hypotheses are assumed. Nevertheless, we will see hereafter that the normal distribution naturally derives form a), b) and c).

In these seminal works Einstein performed the transition density of such a process $P(X_{t+h} \in dy | X_h = x) = q(t, x, y)dy$ and showed that this density is linked to the heat equation (the function $u(t, x) = E[f(X_{t+h})|X_h = x]$ is the unique solution of the ordinary differential equation $-\frac{\partial u(t,x)}{\partial t} + \frac{1}{2}\frac{\partial^2 u(t,x)}{\partial x^2} = 0$ with initial condition $u(0, .) = f$).

A remarkable bridge between probability theory and analysis was build.

From a mathematical point of view, the rigorous proof of the existence of such a stochastic process will appear later (1923) in the works of N.Wiener [16]. Surprisingly, the study of the properties of the "old" Brownian motion, initiated by P. Lévy [12], is still an active field of research. Moreover, the mathematical model of Brownian motion has several real-world applications. An often quoted example is stock market fluctuations. Another example is the evolution of physical characteristics in the fossil record.

Finally, the french mathematician Wendelin Werner has recently obtain (2006) the Fields medal for his entire work on random phenomena. (see for example his lecture "http://www.canalu.fr/canalu/chainev2/utls/programme/324388617/sequence_id/1010501222/format_id/3003/" in the Université de tous les sasees). This is the first time the medal was awarded to a probabilist. It's an acknowledgment of works on random walks and Brownian motion, which model many physical phenomena.

Before technical considerations, let us start with the following result that is important to understand the definition of the Brownian motion that will be given hereafter.

Lemma 1.1.1 *Let X_t be a stochastic process fulfilling assumptions a), b) and c). Suppose that $X_0 = 0$, then , there both exist m and σ^2 such that X_t follows a $\mathcal{N}(mt, \sigma^2 t)$.*

Proof: For simplicity we suppose that

$$\sup_{t \le 1} E[X_t^2] < +\infty$$

(this fact may be derived from the hypotheses as you can see in [8]). In this case, the trajectories of the stochastic process are continuous in L^1. In fact , for $\varepsilon > 0$, we derive from

$$E[|X_t - X_s|] = E[|X_t - X_s| 1_{|X_t - X_s| > \varepsilon}] + E[|X_t - X_s| 1_{|X_t - X_s| \leq \varepsilon}]$$

and from Holder inequality that

$$E[|X_t - X_s|] \leq E[|X_t - X_s|^2]^{\frac{1}{2}} P(|X_t - X_s| > \varepsilon)^{\frac{1}{2}} + \varepsilon.$$

The conclusion follows because the pathwise continuity implies the continuity in probability.

For $n \in \mathbb{N}^*$, we have $X_{nt} = X_t + (X_{2t} - X_t) + ... + (X_{nt} - X_{(n-1)t})$, thus, using b), $\forall n \in \mathbb{N}$,

$$E[X_{nt}] = nE[X_t].$$

For $(p, q) \in \mathbb{N} \times \mathbb{N}^*$, we deduce from the preceding relation that

$$pE[X_1] = E[X_p] = E[X_{(\frac{p}{q})q}] = qE[X_{\frac{p}{q}}],$$

so, $\forall s \in \mathbb{Q}$, $E[X_s] = sE[X_1]$. From the continuity of the trajectories in L^1, we obtain,

$$\forall t \in \mathbb{R}, E[X_t] = tE[X_1] = tm.$$

In the same way, we can prove that

$$E[(X_s - ms)^2] = sE[(X_1 - m)^2], \forall s \in \mathbb{Q},$$

and this equality extends to whole $\mathbb{R}$ because the function $t \to E[(X_t - mt)^2]$ is non-decreasing: using b),

$$E[(X_{t+h} - m(t+h))^2] = E[(X_t - mt)^2] + E[(X_h - mh)^2] \geq E[(X_t - mt)^2].$$

Writing, $\forall n \in \mathbb{N}^*$,

$$X_t = (X_{\frac{t}{n}} - X_0) + ...(X_{\frac{nt}{n}} - X_{\frac{(n-1)t}{n}})$$

where the random variables $(X_{\frac{kt}{n}} - X_{\frac{(k-1)t}{n}})$ are i.i.d with means $\frac{tm}{n}$ and variances $\frac{t\sigma^2}{n}$, a classical argument (the Taylor expansion of the characteristic function used in the proof of the classical C.L.T) gives the result. (see [5]). □

1.2 Definition, existence, simulation

1.2.1 Definition

Let $(\Omega, \mathcal{A}, P)$ be a probability space. To lighten the notations, we work on the bounded interval $[0, T]$ with $T = 1$ (everything remains ok if we take a general T or $\mathbb{R}_+$).

The Brownian motion $(B_t)_{t\in[0,1]}$ is a continuous Gaussian process with independent and stationary increments. In other words:

Definition 1.2.1 *Standard Brownian motion (B.M) is a stochastic process* $(B_t)_{t\in[0,1]}$ *fulfilling :*

a) $B_0 = 0$ *P-a.s.*

b) B is continuous i.e $t \to B_t(w)$ *is continuous for P almost all w.*

c) B has independent increments: For Si $t > s$, $B_t - B_s$ *is independent of* $\mathcal{F}_s^{\mathcal{B}} = \sigma(B_u, u \leq s)$.

d) the increments of B are stationary and gaussian: For $t > s$, $B_t - B_s$ *follows a* $\mathcal{N}(0, t-s)$.

Remark 1.2.1 *a) As mentioned in paragraph 0.5, the filtration* $(\mathcal{F}_t^{\mathcal{B}})_{t\in[0,1]}$ *is supposed to be complete.*

b) From an adaptation of the proof of lemma 1.1.1 we deduce that assumption d) can be replaced by the following "The increments of B are stationary, centered, square integrable with $Var(B_1) = 1$*".*

c) By the monotone class theorem, we show that c) is equivalent to "For $t_1 \leq ... \leq t_n \leq s < t$, $B_t - B_s \perp (B_{t_1}, ..., B_{t_n})$*" (see [4]).*

We have the following equivalent definition linked to the theory of Gaussian processes

Proposition 1.2.1 *A stochastic process* $(B_t)_{t\in[0,1]}$ *is a B.M if and only if it is a continuous and centered gaussian process with covariance function* $\Gamma[s,t] = inf(s,t)$.

Proof: $\boxed{\Rightarrow}$ For arbitrary $t_1 \leq ... \leq t_n$ the random vector $(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})$ is a Gaussian vector since B_{t_1}, $B_{t_2} - B_{t_1}$,..., $B_{t_n} - B_{t_{n-1}}$ are independent Gaussian random variables . Thus, by linear combinations $(B_{t_1}, ..., B_{t_n})$ is also a Gaussian vector. So, B.M is a centered and continuous Gaussian process such that: for $t > s$

$$cov(B_t, B_s) = E[B_t B_s] = E[(B_t - B_s)B_s] + E[(B_s)^2] = s.$$

$\boxed{\Leftarrow}$ We prove each point of the definition

a) $E[(B_0)^2] = 0$ thus $B_0 = 0$ P-a.s.
b) from the hypotheses B is continuous.
c) For $t_1 \leq ... \leq t_n \leq s < t$ the random vector $(B_{t_1}, ...B_{t_n}, B_t - B_s)$ is gaussian with $cov(B_t - B_s, B_{t_i}) = 0$. Thus (corollary and exercise 0.3.1) $B_t - B_s \perp (B_{t_1}, ..., B_{t_n})$ and from the remark above, $B_t - B_s$ is independent of $\mathcal{F}_s^{\mathcal{B}} = \sigma(B_u, u \leq s)$. d) finally, for $s < t$, $B_t - B_s$ follows a centered Gaussian distribution with $Var(B_t - B_s) = t + s - 2inf(s,t) = t - s$. $\square$

1.2.2 Existence, construction, simulation

Several methods, more or less abstract, permit to construct the B.M. In general, these methods need non-trivial mathematical results.

Randomization of an Hilbert space

Let $(g_n)_{n\in\mathbb{N}}$ be i.i.d standard Gaussian random variables. We consider the Hilbert space $H = L^2([0,1], dx)$ and $(\chi_n)_{n\in\mathbb{N}}$ an associated orthonormal basis. We set, $\forall t \in [0,1]$,

$$B_t = \sum_{n=0}^{\infty} \int_0^t \chi_n(s)ds \; g_n$$

(where the serie converges in L^2). According to exercise 0.2.2, the random variable B_t is Gaussian, centered, with variance

$$Var(B_t) = \sum_{n=0}^{\infty} (\int_0^t \chi_n(s)ds)^2 = t.$$

In the same way, we can show easily that $(B_t)_{t\in[0,1]}$ is a Gaussian process with zero mean and covariance function $\Gamma[s,t] = inf(s,t)$. Using proposition 1.2.1, it remains to prove the continuity. A detailed study of the random serie (studying the a.s uniform convergence) is sufficient to conclude but is technical ([13]). Another point of view is to use the powerful Kolmogorov theorem (see [1]):

Theorem 1.2.1 *(Continuity criteria) Let $(X_t)_{t\in\mathbb{R}}$ be a stochastic process fulfilling the following relation, $\forall T > 0$, $\exists C_T \geq 0$, $\forall 0 \leq s < t \leq T$,*

$$E[|X_t - X_s|^p] \leq C_T |t - s|^{\alpha} \tag{1.1}$$

where $p > 0$ and $\alpha > 1$. Then, there exists a version of X with continuous paths.

In the case of the Brownian motion, for $\forall 0 \leq s < t \leq 1$, $B_t - B_s$ follows a $\mathcal{N}(0, t-s)$ thus, from exercise 0.1.1,

$$E[|B_t - B_s|^{2k}] = \frac{(2k)!}{2^k k!}(t-s)^k.$$

The result follows taking for example $k = 2$.

If we explicitly chose $(\chi_n)_{n\in\mathbb{N}}$, we have the two following so-called methods:

Wiener representation (1923) ([16])

Taking for $(\chi_n)_{n\in\mathbb{N}}$ the trigonometric basis, we obtain a closed formula for the B.M based on random Fourier series and discovered by Wiener . From an historical point of view this result is the first rigorous construction of the B.M.

$$B_t = \frac{\sqrt{8}}{\pi} \sum_{n=1}^{\infty} \frac{sin(nt)}{n} \; g_n \tag{1.2}$$

where the serie a.s uniformly converges on $[0, 1]$. Moreover $E[B_t] = 0$ and a classical result gives

$$E[(B_t)^2] = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{sin^2(nt)}{n^2} = t.$$

Paul Lévy construction (midpoint method)

In 1939, Paul Lévy proposed a very simple construction of the Brownian motion taking for $(\chi_n)_{n\in\mathbb{N}}$ the Haar basis. This approach is very important because it has permitted to prove important results concerning the regularity of the Brownian paths. In this paragraph we present the intuitive aspects and we refer the reader to [14] for more details.

For $s < t$ we know that (proposition 1.2.1) the random vector

$$(B_{\frac{t+s}{2}}, B_t, B_s)$$

is Gaussian. Then, we put

$$Z = B_{\frac{t+s}{2}} - \frac{1}{2}(B_t + B_s)$$

that is a Gaussian random variable with $E[Z] = 0$ and $Var(Z) = \frac{1}{4}(t-s)$. Using corollary 0.3.1, we have $Z \perp (B_t, B_s)$ because $cov(Z, B_t) = 0$ and $cov(Z, B_s) = 0$. Thus Z may be written as $Z = \frac{\sqrt{t-s}}{2} G_{s,t}$ where $G_{s,t}$ is a standard Gaussian random variable independent of (B_t, B_s).

Remark 1.2.2 *We can show that* $G_{s,t} \perp B_u$ *when* $u \leq s$ *or* $u \geq t$.

To conclude

$$B_{\frac{t+s}{2}} = \frac{1}{2}(B_t + B_s) + \frac{\sqrt{t-s}}{2} G_{s,t}$$

$$G_{s,t} \perp B_u \text{ when } u \leq s \text{ or } u \geq t.$$

Now we can simulate the Brownian trajectory:

1) We generate a family $(G_i)_{i \in \mathbb{N}}$ of i.i.d standard Gaussian random variables (exercise 0.2.1).

2) We put $B_1 = G_1$

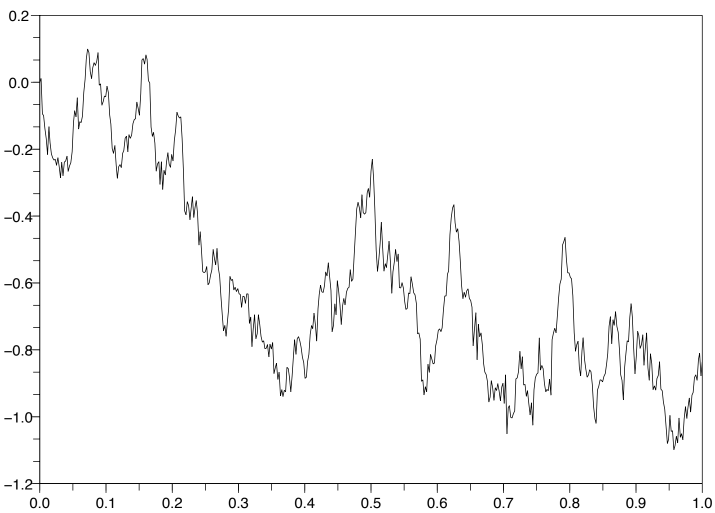
3) We put $B_{\frac{1}{2}} = \frac{1}{2}(B_1 + G_2)$

4) We put $B_{\frac{1}{4}} = \frac{1}{2}(B_{\frac{1}{2}} + \frac{1}{\sqrt{2}} G_3)$

5) We put $B_{\frac{3}{4}} = \frac{1}{2}(B_{\frac{1}{2}} + B_1 + \frac{1}{\sqrt{2}} G_4)$

6) Etc......

Using the software SCILAB we obtain the following:



Donsker invariance principle

The Donsker invariance principle is a functional extension of the C.L.T. Consider a family $(U_k)_{k\in\mathbb{N}^*}$ of centered and independent random variables with variances equal to 1. For $n \in \mathbb{N}^*$, we set $S_n = U_1 + ... + U_n$ the $n-$th partial sum. According to the C.L.T, $\frac{S_n}{\sqrt{n}} \underset{\mathcal{D}}{\to} \mathcal{N}(0,1)$ and more generally, $\forall t \in [0,1]$,

$$\frac{S_{[nt]}}{\sqrt{n}} \underset{\mathcal{D}}{\to} \mathcal{N}(0,t) \text{ (distribution of } B_t).$$

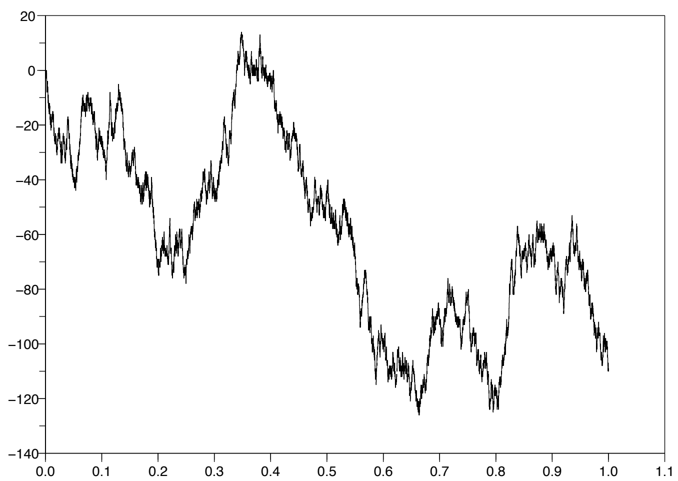
The linear interpolation of the points of the form $(\frac{k}{n}, \frac{S_k}{\sqrt{n}})_{0<k\leq n}$ is defined: $\forall t \in [0,1]$, we put

$$X_n(t) = \frac{1}{\sqrt{n}}\left(\sum_{k=1}^{[nt]} U_k + (nt - [nt])U_{[nt]+1}\right). \tag{1.3}$$

The proof of the following result may be found in [1].

Theorem 1.2.2 *The sequence of continuous stochastic processes* $(X_n)_{n\in\mathbb{N}^*}$ *converges in distribution in the space* $\mathcal{C} = C([0,1],\mathbb{R})$ *towards the Brownian motion:* $\forall f \in C_b(\mathcal{C},\mathbb{R})$, $E[f(X_n)] \to E[f(B)]$.

Taking U_i such that $P(U_i = 1) = P(U_i = -1) = \frac{1}{2}$ and $n = 10000$ we obtain the following simulation.



1.3 Properties

This part presents elementary but important results concerning the B.M. For more details we refer to ([15]). Here, we consider that $(B_t)_{t\geq 0}$ is a B.M $\mathbb{R}_+$.

1.3.1 Martingale property

Proposition 1.3.1 *The stochastic processes* (B_t), $((B_t)^2 - t)$ *and* $(e^{\theta B_t - \theta^2 \frac{t}{2}})$ $(\theta \in \mathbb{R})$ *are* $(\mathcal{F}_t^{\mathcal{B}})_{t\in\mathbb{R}_+}$ *martingales.*

The following proposition is due to Paul Lévy.

Proposition 1.3.2 *Let* $(X_t)_{t\geq 0}$ *be a continuous martingale (with respect to the filtration* $(\mathcal{F}_t^X)_{t\geq 0}$*) starting from* 0*. Then* X *is a B.M if and only if one of the two following conditions is fulfilled:*

a) The process $t \to (X_t)^2 - t$ *is a martingale.*

b)The process $t \to e^{\theta X_t - \theta^2 \frac{t}{2}}$ *is a martingale for all* $\theta \in \mathbb{R}$*.*

Proof: We only prove that a) implies that X_t is a B.M. (the implication "b) $\Rightarrow X_t$ is a B.M" is left to the reader and based on the Itô formula [see chap.2]). For $t > s$, we have $\forall \theta \in \mathbb{R}$,

$$E[e^{i\theta(X_t - X_s)}|\mathcal{F}_s^X] = e^{\theta^2 \frac{t-s}{2}}$$

and taking the expectation

$$E[e^{i\theta(X_t - X_s)}] = e^{\theta^2 \frac{t-s}{2}}.$$

This implies that $X_t - X_s \perp \mathcal{F}_s^X$ because if Y is $\mathcal{F}_s^X$ measurable, $\forall u \in \mathbb{R}$

$$E[e^{i\theta(X_t - X_s) + iuY}] = E[E[e^{i\theta(X_t - X_s) + iuY}|\mathcal{F}_s^X]] = E[e^{iuY}]e^{\theta^2 \frac{t-s}{2}} = E[e^{iuY}]E[e^{i\theta(X_t - X_s)}]$$

and that $X_t - X_s$ follows a $\mathcal{N}(0, t-s)$.$\square$

1.3.2 Transformations

Proposition 1.3.3 *We put* $B_t^{(s)} = B_{t+s} - B_s$*, for fixed* s*,* $Y_t = cB_{\frac{t}{c^2}}$*,* $c > 0$*,* $Z_t = tB_{\frac{1}{t}}$*,* $t > 0$*,* $Z_0 = 0$*. Then, the stochastic processes* $-B_t$*,* $B_t^{(s)}$*,* Y_t *and* Z_t *are standard Brownian motions.*

The fact that Y_t is a B.M is known as the "scaling" property or the "self-similarity' ': No matter what scale you examine Brownian motion on, it looks just the same.

Proof of the proposition: For $-B_t$, $B_t^{(s)}$ and Y_t the proof is obvious. As far as Z_t is concerned, it is easy to prove that I Z_t is a centered Gaussian process with covariance function $\Gamma(s,t) = inf(s,t)$. It remains to show the continuity (the continuity at 0). This result is not easy ([14]) and we restrict ourselves to $\frac{B_n}{n} \underset{n\to\infty}{\to} 0$. Since

$$\frac{B_n}{n} = \frac{(B_n - B_{n-1}) + ... + (B_1 - B_0)}{n}$$

where the $(B_{i+1} - B_i)'s$ are a $n-$sample of the standard normal distribution, we can conclude using the S.L.L.N.□

1.3.3 Regularity of the paths

Proposition 1.3.4 *Let (B_t) be a Brownian motion. Then P-a.s,*

a) $lim\ sup_{t\to+\infty} \frac{B_t}{\sqrt{t}} = lim\ sup_{t\to 0+} \frac{B_t}{\sqrt{t}} = +\infty$

b) $lim\ inf_{t\to+\infty} \frac{B_t}{\sqrt{t}} = lim\ inf_{t\to 0+} \frac{B_t}{\sqrt{t}} = -\infty$

Proof: For a) consider the random variable

$$R = lim\ sup_{t\to+\infty} \frac{B_t}{\sqrt{t}} = lim\ sup_{t\to+\infty} \frac{B_t - B_s}{\sqrt{t}}\ (\forall s \geq 0).$$

By the independence of the Brownian increments $R \perp \sigma(B_u, u \leq s)$ for all $s \geq 0$ thus $R \perp \sigma(B_u, u \geq 0)$. Hence $R \perp R$ and R is a constant (finite or infinite). Suppose that R is finite, thus (remind the definition of $lim\ sup$), $P(\frac{B_t}{\sqrt{t}} \geq R+1) \to 0$ when $t \to +\infty$. But $P(\frac{B_t}{\sqrt{t}} \geq R+1) = P(B_1 \geq R+1) > 0$ the result follows. For the second part of the equality the reasoning is the same.

The point b) is a direct consequence of both a) and the symmetry of the B.M.□

Corollary 1.3.1 *i)* $\forall c \in \mathbb{R}$,

$$P(\{\exists \text{ an infinite number of } t_0 \in [0,T] \text{ such that } B_{t_0} = c\}) = 1.$$

ii) P-a.s, the Brownian sample path (B_t) is nowhere differentiable from the right (resp. nowhere differentiable from the left).

Proof: i) is a direct consequence of both the continuity of sample paths and the preceding proposition.

ii)P-a.s, B_t in not differentiable at 0 from the right because $lim\ sup_{t\to 0+}\frac{B_t - B_0}{\sqrt{t}} = +\infty$. Considering (with the notations of proposition 1.3.3) the transformations B^s_t and Z_t, we show that P-a.s, the Brownian sample path (B_t) is nowhere differentiable from the right (resp. nowhere differentiable from the left).□

Remark 1.3.1 *The sample paths of the B.M are a celebrated example of continuous functions nowhere differentiable. In general, such a function is not easy to build (cf. Weierstrass function) without probability theory. Intuitively, (§1.1), the nowhere differentiability implies that we are not able to measure the speed of a pollen particle. This directly comes from the fact that we have neglected the mass (i.e the inertia) in the preceding model.*

Proposition 1.3.5

$$P(\{\omega; t \to B_t(w) \textit{ is monotone in no interval }\}) = 1$$

Proof: We set $F = \{\omega;$ there exists an interval where $t \to B_t(w)$ is monotone$\}$. We have

$$F = \bigcup_{(s,t)\in\mathbb{Q}^2,\ 0\le s<t} \{\omega; t \to B_t(w) \text{ is monotone on } [s,t]\}.$$

For fixed $0 \le s < t$ in $\mathbb{Q}$, we study for example

$$A = \{\omega; t \to B_t(w) \text{ is nondecreasing on } [s,t]\}.$$

But $A = \bigcap_{n>0}\bigcap_{i=0}^{n-1} A^n_i$ where $A^n_i = \{\omega; B_{s+(t-s)\frac{i+1}{n}} - B_{s+(t-s)\frac{i}{n}} \ge 0\}$. By independence and stationarity,

$$P(\bigcap_{i=0}^{n-1} A^n_i) = \frac{1}{2^n}.$$

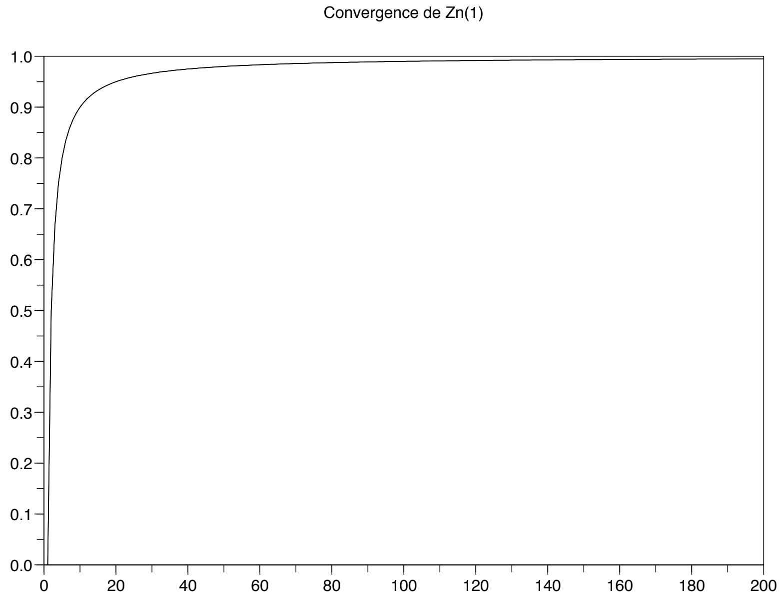
Finally, $\forall n > 0$, $P(A) \le \frac{1}{2^n}$, thus $P(A)$ and $P(F)$ are equal to zero.□

1.3.4 Variation and quadratic variation

The following proposition is fundamental.

Proposition 1.3.6 *For* <u>$t > 0$</u>*, we set* $\forall n \in \mathbb{N}$, $\forall j \in \{0, ..., 2^n\}$, $t^n_j = \frac{tj}{2^n}$*. Then,*

$$Z^n_t = \sum_{j=1}^{2^n} |B_{t^n_j} - B_{t^n_{j-1}}|^2 \underset{a.s \textit{ and } L^2}{\to} t.$$



Proof: We have $E[Z_t^n] = t$ thus in order to prove the convergence in L^2 we only have to show that $Var(Z_t^n) \to 0$. But,

$$Var(Z_t^n) = \sum_{j=1}^{2^n} Var(|B_{t_j^n} - B_{t_{j-1}^n}|^2) = \sum_{j=1}^{2^n} (\frac{t}{2^n})^2 = 2^{-n+1}t^2,$$

the last equality coming from the fact that $E[X^4] = 3\sigma^4$ when $X \sim \mathcal{N}(0, \sigma^2)$. Moreover

$$E\left[\sum_{n=1}^{\infty} |Z_t^n - t|^2\right] < \infty.$$

Thus, using Tchebychev inequality and proposition 0.2.3 the *a.s* is also proved. □

Corollary 1.3.2

$$\sum_{j=1}^{2^n} |B_{t_j^n} - B_{t_{j-1}^n}| \underset{a.s}{\to} +\infty.$$

The Brownian paths are a.s of infinite variation on any interval.

Proof: Suppose that $P(lim_{n\to\infty}\sum_{j=1}^{2^n}|B_{t_j^n}-B_{t_{j-1}^n}|<\infty)>0$. In this case

$$t=lim_{n\to\infty}\sum_{j=1}^{2^n}|B_{t_j^n}-B_{t_{j-1}^n}|^2\leq lim_{n\to\infty}\max_{1\leq j\leq 2^n}|B_{t_j^n}-B_{t_{j-1}^n}|\; lim_{n\to\infty}\sum_{j=1}^{2^n}|B_{t_j^n}-B_{t_{j-1}^n}|.$$

The brownian paths being continuous on $[0,1]$ they are uniformly continuous, thus

$$lim_{n\to\infty}\max_{1\leq j\leq 2^n}|B_{t_j^n}-B_{t_{j-1}^n}|=0\ P-a.s.$$

From $P(lim_{n\to\infty}\sum_{j=1}^{2^n}|B_{t_j^n}-B_{t_{j-1}^n}|<\infty)>0$, we obtain a contradiction because $t>0$.□

1.3.5 Markov property

The following proposition ensures that the B.M is a Markov process i.e a process such that the future states, given the present state and all past states, depends only upon the present state and not on any past states.

Proposition 1.3.7 *For any $f:\mathbb{R}\to\mathbb{R}$ measurable and bounded, for $s<t$,*

$$E[f(B_t)|\mathcal{F}_s^B]=E[f(B_t)|B_s]=\frac{1}{\sqrt{2\pi(t-s)}}\int_{\mathbb{R}}f(x)e^{-\frac{(y-B_s)^2}{2(t-s)}}dy.$$

Proof: Since $f(B_t)=f(B_t-B_s+B_s)$, where $B_t-B_s\perp B_s$ and $B_t-B_s\sim\mathcal{N}(0,t-s)$, the result follows from proposition 0.4.3.□

1.3.6 Geometric Brownian motion

Definition 1.3.1 *For $(b,\sigma)\in\mathbb{R}^2$, the stochastic process*

$$X_t=X_0e^{(b-\frac{1}{2}\sigma^2)t+\sigma B_t}$$

is called a geometric Brownian motion.

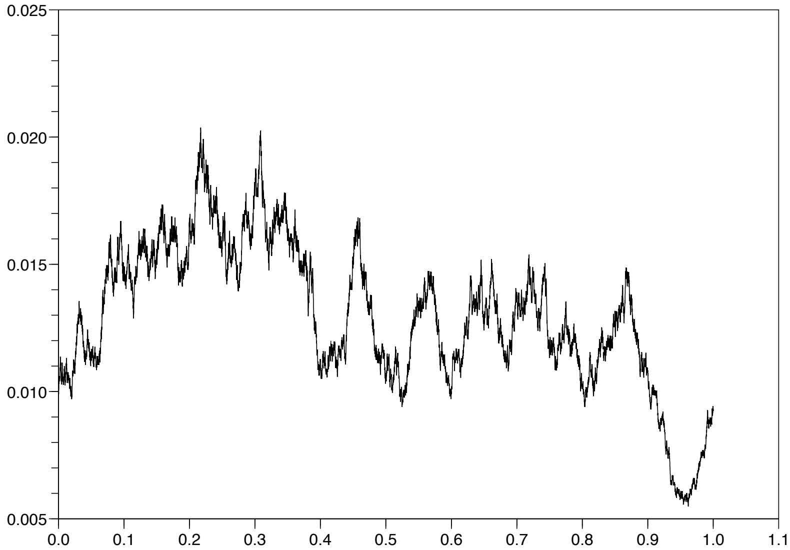
This process is "log-normal": for $X_0=x>0$,

$$ln(X_t)=(b-\frac{1}{2}\sigma^2)t+\sigma B_t+ln(x)$$

has a normal distribution.

Remark 1.3.2 *X being a function of the B.M, it is easy to simulate.*

The following simulation is done for $b=0$, $\sigma=1$, $X_0=1$ and $t\in[0,1]$ (B.M has been simulated using the Donsker method with $n=10000$). Remark that this process is always nonnegative.



Exercise 1.3.1 *a) Show that $X_t e^{-bt}$ is a martingale.*

b) Show that $E[X_t|X_0] = X_0 e^{bt}$ and $Var(X_t|X_0) = X_0^2 e^{2bt}(e^{\sigma^2 t} - 1)$.
c) For $f \in C_b(\mathbb{R}, \mathbb{R})$and $t > s$ show that

$$E[f(X_t)|\mathcal{F}_s^B] = \int_{-\infty}^{+\infty} f(X_s e^{(b-\frac{1}{2}\sigma^2)(t-s+\sigma y\sqrt{t-s})})\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy.$$

(the geometric Brownian motion is a Markov process)

1.3.7 Wiener Integral

In this part we present a first elementary approach of the stochastic integral (i.e an integral with respect to the B.M). We restrict ourselves to determinist integrands (or special stochastic ones).

Reminder on integration theory

We refer the reader to [14] for more details on this subject.

Let $g: [0,1] \to \mathbb{R}$ be a continuous nondecreasing function (actually continuous from the right is sufficient...) such that $g(0) = 0$. A classical result ensures the existence of a bounded measure m on $([0,1],\mathcal{B}([0,1]))$ such that $g(t) = m([0,t])$ (if g is nonnegative, the measure m is nonnegative and g the associated distribution function). Now, it is easy to define the integral with respect to g putting $\forall f \in L^1(m)$,

$$\int_0^t f(s)dg(s) = \int 1_{[0,t]} f dm.$$

This construction may be extended to less regular functions:

Definition 1.3.2 *A function g $[0,1] \to \mathbb{R}_+$ is said to be of bounded variations if*

$$sup \sum_{j=1}^{n} |f(t_j) - f(t_{j-1})| < \infty,$$

where the supremum is over partitions $t_0 = 0 \leq t_1 \leq ..... \leq t_n = 1$ of the interval $[0,1]$.

Remark 1.3.3 *If g is differentiable, g has bounded variations and in this case* $\int_0^t f(s)dg(s) = \int_0^t f(s)g'(s)ds$.

Proposition 1.3.8 *When g has bounded variations, $g(0) = 0$ and g is continuous, there exist two continuous nondecreasing functions g_1 and g_2 with $g_1 = g_2 = 0$ such that $g = g_1 - g_2$. Thus the integral with respect to g is built from the integrals with respect to g_1 and g_2.*

According to corollary 1.3.2, we can see that for P almost all ω, the Brownian sample paths are of infinite variations. It is not possible to build the stochastic integral ω by ω.

Nevertheless we will adopt another point of view.

Construction of the Wiener integral

A method is to generalize the construction of the B.M by randomization of an Hilbert space.

Let $(g_n)_{n\in\mathbb{N}}$ be i.i.d standard Gaussian random variables. We consider the Hilbert space $H = L^2([0,1], dx)$ and $(\chi_n)_{n\in\mathbb{N}}$ an associated orthonormal basis. We set, $\forall f \in H$,

$$I(f)_t = \sum_{n=0}^{\infty} \int_0^t f(s)\chi_n(s)ds \; g_n \tag{1.4}$$

(the serie being convergent in L^2). Since $B_t = \sum_{n=0}^{\infty} \int_0^t \chi_n(s)ds \; g_n$, $I(f)_t$ will be denoted by $\int_0^t f(s)dB_s$. From exercise 0.2.2, $\int_0^t f(s)dB_s$ is centered Gaussian random variable with variance $\int_0^t f^2(s)ds$. The continuity of the process $(\int_0^t f(s)dB_s)$ can be prove directly by technical arguments or simply using theorem 1.2.1.

The proof of the following is left to the reader as an exercise.

Proposition 1.3.9 *Properties of the Wiener integral*

a) $f \in H \to \int_0^t f(s)dB_s$ *is linear.*

b) $(\int_0^t f(s)dB_s)_{t\in[0,1]}$ *is a continuous and centered Gaussian process with covariance function* $\Gamma(s,t) = \int_0^{\inf(s,t)} f^2(u)du$.

c) $(\int_0^t f(s)dB_s)_{t\in[0,1]}$ *is adapted with respect to* $(\mathcal{F}_t^B)_{t\in[0,1]}$ *with independent increments (but no stationarity).*

d) When $(f,g) \in H^2$, $E\left[\int_0^t f(s)dB_s \int_0^u g(s)dB_s\right] = \int_0^{\inf(t,u)} f(s)g(s)ds$.

e) $\left(\int_0^t f(s)dB_s\right)_{t\in[0,1]}$ *and* $\left((\int_0^t f(s)dB_s)^2 - \int_0^t f^2(s)ds\right)_{t\in[0,1]}$ *are* $(\mathcal{F}_t^B)_{t\in[0,1]}$ *martingales.*

f) $(\int_0^t f(s)dB_s)_{t\in[0,1]}$ *fulfills the Markov property.*

When f is regular, the Wiener integral is actually defined ω by ω.

Proposition 1.3.10 *When* $f \in C^1([0,1],\mathbb{R})$, *then,* $\forall 1 \geq t \geq 0$,

$$\int_0^t f(s)dB_s = f(t)B_t - \int_0^t f'(s)B_s ds.$$

Proof: We use (1.4) and the following equality

$$\int_0^t f(s)\chi_n(s)ds = -\int_0^t \left(\int_s^t f'(u)du\right)\chi_n(s)ds + f(t)\int_0^t \chi_n(s)ds.$$

□

Remark 1.3.4 *This neat construction of the Wiener integral has a principal defect: In fact, we can't see cleary what are the essential properties of the Browian motion that make it possible. We will see in the sequel that the orthogonality of the Brownian increments is the key stone of such a construction and will allow us to extend this procedure to general integrands.*

Exercise 1.3.2 ***A stochastic version of Fubini theorem***
Let $f : [0,1]^2 \to \mathbb{R}$ be a continuous mapping, the aim of the exercise is to prove the following formulale

$$\int_0^1 \int_0^1 f(s,t) dB_s \, dt = \int_0^1 \int_0^1 f(s,t) dt \, dB_s.$$

a) Show that the integrals mentioned above are well defined.
b) Show that

$$\int_0^1 \left(\sum_{n=0}^N \int_0^1 f(s,t) \chi_n(s) ds \; g_n \right) dt = \sum_{n=0}^N \int_0^1 \left(\int_0^1 f(s,t) dt \right) \chi_n(s) ds \; g_n.$$

Conclude (letting n goes to infinity).

Integrands of the form $f(B_t)$

The following proposition gives a definition of the stochastic integral for particular stochastic integrands. This approach is the stochastic counterpart to the construction of the Lebesgue integral by Riemann sums.

Proposition 1.3.11 *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with bounded derivative. Then, the serie*

$$Z_n = \sum_{i=0}^{n-1} f(B_{\frac{ti}{n}})(B_{\frac{t(i+1)}{n}} - B_{\frac{ti}{n}}) \tag{1.5}$$

converges in L^2 and we denote by $\int_0^t f(B_s) dB_s$ its limit (in general, this limit doesn't follow a normal distribution).

Proof: The proof will be given in the sequel.□

Remark 1.3.5 *When f is regular, the Riemann theorem ensures that $\sum_{i=0}^{n-1} f(x_n^i)(\frac{t(i+1)}{n} - \frac{ti}{n})$ converges toward $\int_0^t f(s) ds$ if $x_n^i \in [\frac{ti}{n}, \frac{t(i+1)}{n}]$. Thus, we have the choice of the position of x_n^i in the intervall $[\frac{ti}{n}, \frac{t(i+1)}{n}]$. As far as the stochastic integral defined in the preceding proposition is concerned, this property is not fulfilled anymore. The choice of $f(B_{\frac{ti}{n}})$ is not innocent (in particular this random variable is independent of $(B_{\frac{t(i+1)}{n}} - B_{\frac{ti}{n}})$). Other choices imply other integrals. For example, taking $f\left(\frac{B_{\frac{t(i+1)}{n}} + B_{\frac{ti}{n}}}{2}\right)$ we obtain the so-called Stratonovitch integral. When $f = Id$ the Stratonovitch integral (that is the limit of $\sum_{i=0}^{n-1} \frac{1}{2}(B_{\frac{t(i+1)}{n}} + B_{\frac{ti}{n}})(B_{\frac{t(i+1)}{n}} - B_{\frac{ti}{n}})$) is equal to $\frac{B_t^2}{2}$. According to the example mentionned below, the limit is different from $\int_0^t B_s dB_s$.*

Comments

Let us consider a stochastic process (G_t) with sample paths of class C^1 and $F : \mathbb{R} \to \mathbb{R}$ a function of class C^1. The classical rules of differential calculus imply that

$$F(G_t) = F(G_0) + \int_0^t F'(G_s)G'_s ds.$$

Moreover, this result may be easily extended to continuous processes of bounded variations.

We know that the Brownian motion has infinite variations (corollary 1.3.2) thus the above formula is not valid as we can see in the following example.

Example 1.3.1 *We want to compute* $\int_0^t B_s dB_s$ *that is the limit in* L^2 *of* $\sum_{i=0}^{n-1} B_{\frac{ti}{n}}(B_{\frac{t(i+1)}{n}} - B_{\frac{ti}{n}})$. *But*

$$2\sum_{i=0}^{n-1} B_{\frac{ti}{n}}(B_{\frac{t(i+1)}{n}} - B_{\frac{ti}{n}}) = \sum_{i=0}^{n-1}(B^2_{\frac{t(i+1)}{n}} - B^2_{\frac{ti}{n}}) - \sum_{i=0}^{n-1}(B_{\frac{t(i+1)}{n}} - B_{\frac{ti}{n}})^2$$

so

$$2\sum_{i=0}^{n-1} B_{\frac{ti}{n}}(B_{\frac{t(i+1)}{n}} - B_{\frac{ti}{n}}) = B_t^2 - \sum_{i=0}^{n-1}(B_{\frac{t(i+1)}{n}} - B_{\frac{ti}{n}})^2.$$

From proposition 1.3.6, the last term converges toward t *in* L^2*, thus*

$$2\int_0^t B_s dB_s = B_t^2 - t.$$

1.3.8 Itô formula for B.M

Proposition 1.3.12 *Consider* $f \in C^2(\mathbb{R}, \mathbb{R})$ *with bounded second derivative. Then,* $\forall t \in [0,1]$,

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds \ P - as.$$

Often, we will use the following differential notation

$$df(B_s) = f'(B_s)dB_s + \frac{1}{2}f''(B_s)ds.$$

Proof: According to the definition mentioned in proposition 1.3.11,

$$\int_0^t f'(B_s)dB_s = \lim_{L^2} Z_n = \lim_{L^2} \sum_{i=0}^{n-1} f'(B_{\frac{ti}{n}})(B_{\frac{t(i+1)}{n}} - B_{\frac{ti}{n}}).$$

Moreover,

$$f(B_t) - f(B_0) = \sum_{i=0}^{n-1} (f(B_{\frac{t(i+1)}{n}}) - f(B_{\frac{ti}{n}}))$$

and, from the Riemann sums theorem,

$$\int_0^t f''(B_s)ds = \lim_{as \text{ and } L^2} \sum_{i=0}^{n-1} f''(B_{\frac{ti}{n}})(\frac{t(i+1)}{n} - \frac{ti}{n}). \tag{1.6}$$

Using a Taylor approximation of order 2 and the continuity of the trajectories,

$$f(B_t) - f(B_0) = \sum_{i=0}^{n-1} (f(B_{\frac{t(i+1)}{n}}) - f(B_{\frac{ti}{n}}))$$

with

$$f(B_{\frac{t(i+1)}{n}}) - f(B_{\frac{ti}{n}}) = f'(B_{\frac{ti}{n}})(B_{\frac{t(i+1)}{n}} - B_{\frac{ti}{n}}) + \frac{1}{2} f''(B_{\alpha_i})(B_{\frac{t(i+1)}{n}} - B_{\frac{ti}{n}})^2$$

where α_i is a random variable with values in $]\frac{ti}{n}, \frac{t(i+1)}{n}[$.

Now, it remains to show that

$$\lim_{L^1} \sum_{i=0}^{n-1} f''(B_{\alpha_i})(B_{\frac{t(i+1)}{n}} - B_{\frac{ti}{n}})^2 = \int_0^t f''(B_s)ds$$

in order to conclude using the unicity of the limits in L^1 (remind that the convergence in L^2 implies the convergence in L^1). This is proved in two steps:

On the first hand, using Schwartz's inequality,

$$E\left[\left|\sum_{i=0}^{n-1} \left(f''(B_{\alpha_i}) - f''(B_{\frac{ti}{n}})\right)(B_{\frac{t(i+1)}{n}} - B_{\frac{ti}{n}})^2\right|\right] \leq U_n V_n$$

with

$$U_n = E\left[sup_i \left|f''(B_{\alpha_i}) - f''(B_{\frac{ti}{n}})\right|^2\right]^{\frac{1}{2}}$$

and

$$V_n = E\left[\left|\sum_{i=0}^{n-1} (B_{\frac{t(i+1)}{n}} - B_{\frac{ti}{n}})^2\right|^2\right]^{\frac{1}{2}}.$$

From proposition 1.3.6, $V_n \to t$. Moreover, $U_n \to 0$ by dominated convergence because the function $s \to f''(B_s)$ is almost surely uniformely continuous on $[0, 1]$ and bounded.

On the other hand, we put

$$W_n = E\left[\left|\sum_{i=0}^{n-1} f''(B_{\frac{ti}{n}})\left(\left(B_{\frac{t(i+1)}{n}} - B_{\frac{ti}{n}}\right)^2 - \left(\frac{t(i+1)}{n} - \frac{ti}{n}\right)\right)\right|^2\right]$$

and using the properties of the Brownian increments we obtain

$$W_n = \sum_{i=0}^{n-1} E\left[\left|f''(B_{\frac{ti}{n}})\left(\left(B_{\frac{t(i+1)}{n}} - B_{\frac{ti}{n}}\right)^2 - \left(\frac{t(i+1)}{n} - \frac{ti}{n}\right)\right)\right|^2\right].$$

Thus

$$W_n \leq \|f''\|_\infty^2 \sum_{i=0}^{n-1} Var((B_{\frac{t(i+1)}{n}} - B_{\frac{ti}{n}})^2) = 2\|f''\|_\infty^2 \frac{t^2}{n} \to 0.$$

According to (1.6), the result follows.□

Exercise 1.3.3 *(difficult...) Let $f \in C^2(\mathbb{R})$ such that*

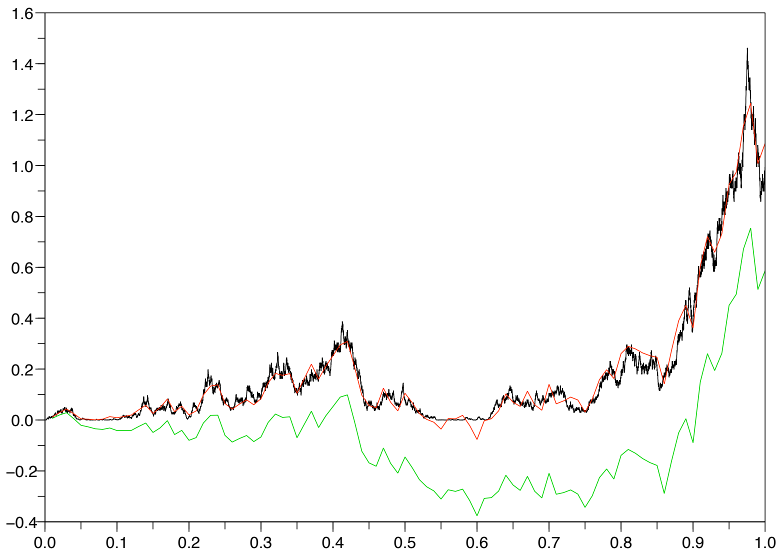
$$E\left[\int_0^T (f'(B_s))^2\, ds\right] < +\infty. \quad (*)$$

Show that $\forall t \in [0,T]$,

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds \;\; P-as.$$

This condition is less restrictive than the preceding one and allow us to apply Itô formula taking for f the exponential mapping.

We can see on the following graph the necessity of a specific differential calculus for the Brownian motion. We have represented in black a trajectory B_t^2 has been represented (simulated using Donsker's approximation), in green a trajectory of $2\int_0^t B_s dB_s$ (simulated using the definition of proposition 1.3.11) and in red a trajectory of $2\int_0^t B_s dB_s + t$.



1.3.9 Applications of the Wiener integral

The Ornstein-Uhlenbeck process

At the beginning of this chapter, we have introduced the B.M to modelize the motion (due to thermal agitation) of small pollen particles floating in water. We have also seen that this model neglect the mass of the particle (i.e the inertia) implying that the trajectories are nowhere differentiable. The following model, proposed by Langevin, is more realistic ([10]).

Consider two forces exerted on a particle of mass m and velocity $V(t)$:

a) a viscous force which is proportional to the particle velocity, $f = -kV$ (Stokes' law) where k is a nonnegative constant linked to the radius of the particle,

b) a complementary force η representing the effect of a continuous series of collisions with the atoms of the underlying fluid and described by Langevin: " elle est indifféremment positive et négative, et sa grandeur est telle qu'elle maintient l'agitation de la particule que, sans elle, la résistance visqueuse finirait par arrêter " .

According to the Newton's second law,

$$mdV(t) = -kV(t)dt + \eta(t)dt. \tag{1.7}$$

The term $\eta(t)dt$ represents the variation of the movement quantity $dM(t)$ between t and $t+dt$. Suppose that

a) $dM(t) = M(t+dt) - M(t)$ is independent of $\sigma(M_s; s \leq t)$.
b) the distribution of $dM(t)$ doesn't depend on t.
c) $t \to M(t)$ is continuous
d) (cf citation) $E[M(t)] = 0$.

Thus we can put $M(t) = \sigma B_t$ and equation (1.7) becomes

$$mdV(t) = -kV(t)dt + \sigma dB_t. \tag{1.8}$$

We have the following definition

Definition 1.3.3 *The Ornstein-Uhlenbeck process is the solution of the following stochastic differential equation*

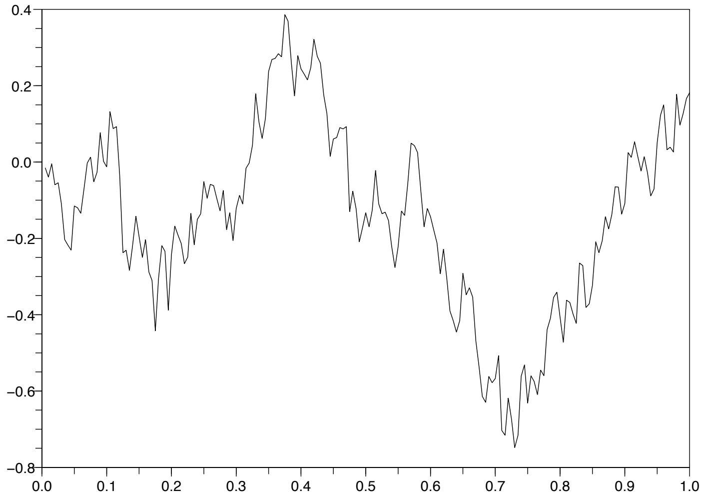
$$X_t = X_0 - a\int_0^t X_s ds + \sigma B_t \tag{1.9}$$

where $(a,\sigma) \in \mathbb{R}^2$ *and* X_0 *are random variables independent of the B.M.*

Proposition 1.3.13 *The equation* (1.9) *has a unique solution given by the following continuous stochastic process:*

$$X_t = e^{-ta}\left(X_0 + \sigma\int_0^t e^{as} dB_s\right). \tag{1.10}$$

The following graph is a simulation of a trajectory of X_t with $a = \sigma = 1$ and $X_0 = 0$.



Proof of proposition 1.3.13: According to proposition 1.3.10,

$$X_t = e^{-ta}\left(X_0 + \sigma e^{at} B_t - \sigma a \int_0^t e^{as} B_s ds\right).$$

Using a stochastic version of Fubini theorem (exercise 1.3.1) we obtain

$$a\int_0^t X_s ds = aX_0 \int_0^t e^{-as} ds + a\sigma \int_0^t B_s ds - a^2\sigma \int_0^t e^{-as}\left(\int_0^s e^{au} B_u du\right) ds.$$

Obvious computations ensure that

$$a\int_0^t X_s ds = X_0 - X_t + \sigma B_t,$$

the existence is proved.

For the unicity, if X_1 and X_2 are solutions of (1.9), then the process $Z = X_1 - X_2$ satisfies

$$Z_t = -a\int_0^t Z_s ds.$$

We may conclude using the following lemma:

Lemma 1.3.1 *(Gronwall) Let $T \in \mathbb{R}^+$, $K \in \mathbb{R}^+$, $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be such that* $\forall t \in [0, T]$

$$\phi(t) \leq K + \int_0^t \phi(s)\psi(s)ds < \infty$$

and

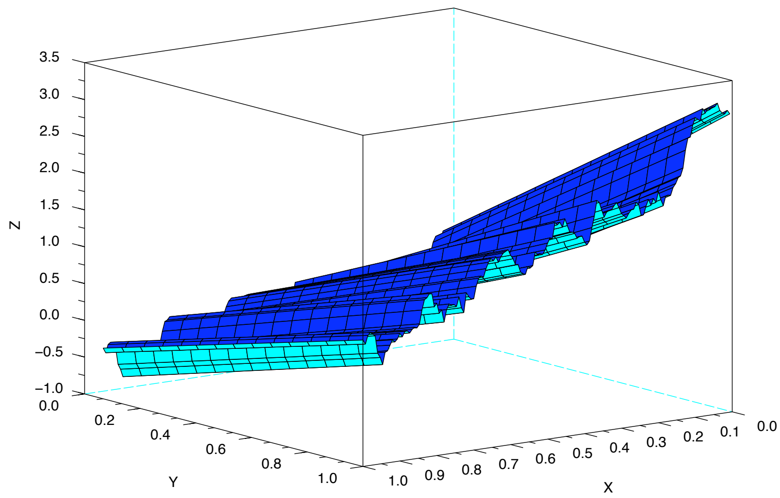
$$\int_0^T \psi(s)ds < \infty.$$

Then,

$$\phi(t) \leq K e^{\int_0^t \psi(s)ds}.$$

□

The following graph represents the flow of the O.U process i.e the application $x_0 \in [0,1] \rightarrow (X_t^{x_0}(\omega))_{t\in[0,1]}$ where ω is fixed and X^{x_0} is the O.U process with initial condition $x_0 \in [0,1]$.



When the initial condition has a normal distribution, we have the following result easily derived from proposition 1.3.9.

Proposition 1.3.14 *Suppose that $X_0 \sim \mathcal{N}(m, \sigma_0^2)$ is independent of the B.M. Then (X_t) is a Gaussian process with mean e*

$$E[X_t] = me^{-at}$$

and covariance function

$$cov(X_s, X_t) = e^{-a(t+s)} \left(\sigma_0 + \frac{\sigma^2}{2a} \left(e^{2a\ inf(s,t)} - 1 \right) \right).$$

Vasicek process

The vasicek process is a slight variant of the OU process with an additional drift. In Finance, it is used as a mathematical "one factor model" describing the evolution of interest rates (cf. [11]). Here the aim is not to introduce the financial theory but to familiarize the reader with underlying computations.

Let Y_t be the stochastic process fulfilling

$$dY_t = a(b - Y_t)dt + \sigma dB_t.$$

We can remark easily that $X_t = Y_t - b$ fulfills equation (1.9) Thus,

$$Y_t = e^{-ta}(Y_0 - b) + b + \sigma \int_0^t e^{-(at-s)} dB_s.$$

We have the same result as in proposition 1.3.14: if $Y_0 \sim \mathcal{N}(m, \sigma_0^2)$ is independent of the B.M, Y is a Gaussian process with mean

$$E[X_t] = me^{-at} + b(1 - e^{-at})$$

and covariance function

$$cov(Y_s, Y_t) = e^{-a(t+s)} \left(\sigma_0 + \frac{\sigma^2}{2a} \left(e^{2a\ inf(s,t)} - 1 \right) \right).$$

Exercise 1.3.4 ***Price of a zero-coupon***

We want to compute $\forall 0 \leq t \leq T$,

$$P(t,T) = E\left[e^{\int_t^T Y_u du} | \mathcal{F}_t^B \right].$$

Suppose that $Y_0 \sim \mathcal{N}(m, \sigma_0^2)$ *is independent of the B.M.*

a) Using exercise 1.3.1, show that

$$Z_t := \int_0^t Y_u du = \frac{1}{a} \left(b(at - 1 - e^{-at}) + Y_0(1 - e^{-at}) + \sigma \int_0^t (1 - e^{-a(t-u)}) dB_u \right).$$

b) Using proposition 0.4.3, compute $E[e^{iu(Z_t - Z_s)}]$ *for* $0 \leq s \leq t \leq T$ *and* $u \in \mathbb{R}$.

c) Deduce from the preceding questions that the conditional distribution of $Z_t - Z_s$ *given* $\mathcal{F}_t^B$ *is gaussian with mean*

$$M(s,t) = b(t-s) + a^{-1}(Y_s - b)(1 - e^{-a(t-s)})$$

and variance

$$V(s,t) = \frac{\sigma_0^2}{a^2}(1-e^{-a(t-s)})^2 + \frac{\sigma^2}{a^2}\left((t-s) + \frac{(1-e^{-2a(t-s)})}{2a} - \frac{2(1-e^{-a(t-s)})}{a}\right).$$

d) Show that

$$P(t,T) = e^{-M(t,T)+\frac{1}{2}V(t,T)}.$$

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Chapter 2

Stochastic integral, Itô processes.

In this chapter $(B_t)_{t\in[0,T]}$ will be a standard brownian motion defined on a probability space $(\Omega, \mathcal{A}, P)$. Moreover this probability space is equipped with the Brownian filtration $(\mathcal{F}_t^B)_{t\in[0,T]}$.

Now we want to extend the construction of Wiener integral in order to be able to integrate regular stochastic processes. The remarkable properties of the Brownian motion will lead to a simple and coherent construction on $\mathcal{E}([0,T]\times\Omega)$ and to an extension to integrand in $L^2_{prog}(\Omega\times[0,T])$ using an argument of isometry. The price to be paid for this simplicity will be the capacity to only integrate adapted processes but it will be enough for elementary financial issues.

2.1 The stochastic integral on $\mathcal{E}([0,T]\times\Omega)$

Consider

$$X_t = \sum_{i=1}^{n-1} F_{t_i} 1_{[t_i,t_{i+1}[(t)} \tag{2.1}$$

in $\mathcal{E}([0,T]\times\Omega)$ (one has $0 \leq t_1 \leq ... \leq t_n \leq T$ and $F_{t_i} \in L^2(\mathcal{F}_{t_i})$).

2.1.1 Definition

Definition 2.1.1 *In the case of an elementary integrand the stochastic integral is given by*

$$\int_0^T X_s dB_s = \sum_{i=1}^{n-1} F_{t_i}(B_{t_{i+1}} - B_{t_i}).$$

Remark 2.1.1 *For* $0 \leq t \leq T$ *one defines naturally*

$$\int_0^t X_s dB_s = \int_0^T X_s 1_{[0,t]}(s) dB_s = \sum_{i=1}^{n-1} F_{t_i}(B_{(t_{i+1}\wedge t)} - B_{(t_i\wedge t)}). \tag{2.2}$$

Of course, when X is a step function (i.e F_{t_i} is constant) this definition coincides with the Wiener one (cf. (1.4)). Nevertheless, it is important to notice that, in general, the stochastic process $\int_0^t X_s dB_s$ is no more Gaussian.

2.1.2 Properties of the Itô integral of an elementary process

Proposition 2.1.1 *On $\mathcal{E}([0,T]\times\Omega)$ the stochastic integral fulfills the following properties:*

a) $X \mapsto \int_0^t X_s dB_s$ is linear.

b) The process $(\int_0^t X_s dB_s)_{t\in[0,T]}$ is continuous.

c) $(\int_0^t X_s dB_s)_{t\in[0,T]}$ is an adapted process with respect to $(\mathcal{F}_t^B)_{t\in[0,T]}$.

d) $E\left[\int_0^t X_s dB_s\right] = 0$ and $Var\left(\int_0^t X_s dB_s\right) = E\left[\int_0^t X_s^2 ds\right]$.

e) For $0 \le s \le t \le T$,

$$E\left[\int_s^t X_u dB_u|\mathcal{F}_s^B\right] = 0 \quad and \quad E\left[\left(\int_s^t X_u dB_u\right)^2 |\mathcal{F}_s^B\right] = E\left[\int_s^t X_u^2 du|\mathcal{F}_s^B\right]. \tag{2.3}$$

f) $(\int_0^t X_s dB_s)_{t\in[0,T]}$ is a continuous and square integrable martingale with respect to $(\mathcal{F}_t^B)_{t\in[0,T]}$ moreover

$$E\left[\sup_{0\le t\le T}|\int_0^t X_s dB_s|^2\right] \le 4E\left[\int_0^T X_u^2 du\right]. \tag{2.4}$$

Proof of the proposition: a) is obvious, b) and c) directly come from(2.2) and d) from e) (taking $s=0$). So we have to show e): Without loss of generality we can assume that (adding two points to the subdivision) $s = t_j$ and $t = t_k$ $(k \ge j)$. Thus,

$$\begin{aligned}
E\left[\int_0^t X_u dB_u|\mathcal{F}_s^B\right] &= E\left[\sum_{i=1}^{k-1} F_{t_i}(B_{t_{i+1}} - B_{t_i})|\mathcal{F}_{t_j}^B\right] \\
&= \sum_{i=1}^{j-1} E\left[F_{t_i}(B_{t_{i+1}} - B_{t_i})|\mathcal{F}_{t_j}^B\right] + \sum_{i=j}^{k-1} E\left[F_{t_i}(B_{t_{i+1}} - B_{t_i})|\mathcal{F}_{t_j}^B\right] \\
&= \sum_{i=1}^{j-1} F_{t_i}(B_{t_{i+1}} - B_{t_i}) + \sum_{i=j}^{k-1} E\left[F_{t_i} E\left[(B_{t_{i+1}} - B_{t_i})|\mathcal{F}_{t_i}^B\right]|\mathcal{F}_{t_j}^B\right] \\
&= \int_0^s X_u dB_u + 0.
\end{aligned}$$

For the second point, the calculus is performed in the same way:

$$E\left[\left(\int_s^t X_u dB_u\right)^2 |\mathcal{F}_s^B\right] = E\left[\left(\sum_{i=j}^{k-1} F_{t_i}(B_{t_{i+1}} - B_{t_i})\right)^2 |\mathcal{F}_{t_j}^B\right]$$
$$= \sum_{i=j}^{k-1} E\left[F_{t_i}^2(B_{t_{i+1}} - B_{t_i})^2|\mathcal{F}_{t_j}^B\right] + 2\sum_{j\leq i<l\leq k-1} E\left[F_{t_l}F_{t_i}(B_{t_{i+1}} - B_{t_i})(B_{t_{l+1}} - B_{t_l})|\mathcal{F}_{t_j}^B\right].$$

Using for the first sum that $E[.|\mathcal{F}_{t_j}^B] = E[E[.|\mathcal{F}_{t_i}^B]|\mathcal{F}_{t_j}^B]$ and for the second one that $E[.|\mathcal{F}_{t_j}^B] = E[E[.|\mathcal{F}_{t_l}^B]|\mathcal{F}_{t_j}^B]$, one has

$$\begin{aligned} E\left[\left(\int_s^t X_u dB_u\right)^2 |\mathcal{F}_s^B\right] &= \sum_{i=j}^{k-1} E\left[F_{t_i}^2(t_{i+1} - t_i)|\mathcal{F}_{t_j}^B\right] + 0 \\ &= E\left[\int_s^t X_u^2 du|\mathcal{F}_s^B\right]. \end{aligned}$$

The point f) is a consequence of e) and of theorem 0.5.3.□

Exercise 2.1.1 *a) For X and Y in $\mathcal{E}([0,T]\times\Omega)$ and for $v, t \geq s$ show that*

$$E\left[\left(\int_s^t X_u dB_u\right)\left(\int_s^v Y_u dB_u\right)|\mathcal{F}_s^B\right] = E\left[\int_s^{t\wedge v} X_u Y_u du|\mathcal{F}_s^B\right].$$

b) show that $\left(\int_0^t X_u dB_u\right)^2 - \int_0^t X_u^2 du$ is a $(\mathcal{F}_t^B)_{t\in[0,T]}$ martingale.

Remark 2.1.2 *The property d) from which we deduce*

$$Var\left(\int_0^T X_s dB_s\right) = E\left[\left(\int_0^T X_u dB_u\right)^2\right] = E\left[\int_0^T X_s^2 ds\right] \tag{2.5}$$

*is fundamental. It implies that the function $X \mapsto \int_0^T X_s dB_s$ is an **isometry** from $\mathcal{E}([0,T]\times\Omega)$ into the space of continuous and square integrable martingales on $[0,T]$, (from now on, this space will be denoted by $M^2([0,T])$).*

2.2 Extension to $L^2_{prog}(\Omega \times [0,T])$

By a standard method (extension of uniformly continuous functions with values in a complete space) it is possible to extend the integral defined above for integrands in $L^2_{prog}(\Omega \times [0,T])$.

Remind that theorem 0.5.2 ensures **the density** of $\mathcal{E}([0,T]\times\Omega)$ in $L^2_{prog}(\Omega \times [0,T])$.

Proposition 2.2.1 *Let $X \in L^2_{prog}(\Omega \times [0,T])$. If Φ_n and Φ'_n are two sequences of elements of $\mathcal{E}([0,T] \times \Omega)$ that converge toward X in $L^2_{prog}(\Omega \times [0,T])$ then*

$$lim \int_0^{\cdot} \Phi_n(.,s)dB_s \underset{M^2([0,T])}{=} lim \int_0^{\cdot} \Phi'_n(.,s)dB_s.$$

Proof: One has to prove two things. First, if Φ_n converges in $L^2_{prog}(\Omega \times [0,T])$ we want to show the convergence $\int_0^{\cdot} \Phi_n(.,s)dB_s$ in $M^2([0,T])$. Since $M^2([0,T])$ is complete (proposition 0.5.6), we only have to prove that $\int_0^{\cdot} \Phi_n(.,s)dB_s$ is a Cauchy sequence in $M^2([0,T])$. This fact comes from (2.5). Secondly, the fact that the limit is independent of the approximating sequence may be shown easily using again (2.5).□

Now we have the following unambiguous definition of the stochastic integral for an integrand $X \in L^2_{prog}(\Omega \times [0,T])$.

Definition 2.2.1 *If $X \in L^2_{prog}(\Omega \times [0,T])$ and if Φ_n is a sequence in $\mathcal{E}([0,T] \times \Omega)$ that converges toward X in $L^2_{prog}(\Omega \times [0,T])$, we denote $(\int_0^{\cdot} X_s dB_s)$ the limit of $(\int_0^{\cdot} \Phi_n(.,s)dB_s)$ in $M^2([0,T])$*

Remark 2.2.1 *We have globally defined the stochastic process $(\int_0^{\cdot} X_s dB_s)$ (instead of t by t). This implies immediately that the limit is a continuous martingale. Note that this integral is defined up to stochastic equivalence.*

A good training is to show that proposition 2.1.1 remains valid in this setting.

Proposition 2.2.2 *On $L^2_{prog}(\Omega \times [0,T])$ the stochastic integral fulfills the following properties:*

a) $X \mapsto \int_0^t X_s dB_s$ is linear.

b) The process $(\int_0^t X_s dB_s)_{t\in[0,T]}$ is continuous.

c) $(\int_0^t X_s dB_s)_{t\in[0,T]}$ is an adapted process with respect to $(\mathcal{F}_t^B)_{t\in[0,T]}$.

d) $E\left[\int_0^t X_s dB_s\right] = 0$ and $Var\left(\int_0^t X_s dB_s\right) = E\left[\int_0^t X_s^2 ds\right]$.

e) For $0 \le s \le t \le T$,

$$E\left[\int_s^t X_u dB_u | \mathcal{F}_s^B\right] = 0 \quad and \quad E\left[\left(\int_s^t X_u dB_u\right)^2 | \mathcal{F}_s^B\right] = E\left[\int_s^t X_u^2 du | \mathcal{F}_s^B\right]. \tag{2.6}$$

f) $(\int_0^t X_s dB_s)_{t\in[0,T]}$ *is a continuous and square integrable martingale with respect to* $(\mathcal{F}_t^B)_{t\in[0,T]}$ *moreover*

$$E\left[\sup_{0\leq t\leq T}|\int_0^t X_s dB_s|^2\right] \leq 4E\left[\int_0^T X_u^2 du\right]. \tag{2.7}$$

Remark 2.2.2 *The stochastic integral coincides not only with the Wiener integral but also with the one define in proposition 1.3.11 (or in exercise 1.3.3 but it is more technical...). In Fact, in the framework of proposition 1.3.11, we only have to show that the process* $Z_n = \sum_{i=0}^{n-1} f(B_{\frac{Ti}{n}})1_{]\frac{Ti}{n},\frac{T(i+1)}{n}]}$ *converges toward* $f(B)$ *in* $L^2_{prog}(\Omega\times[0,T])$*. But*

$$\begin{aligned}
E\left[\int_0^T (f(B_s)-Z_n(s))^2 ds\right]^{\frac{1}{2}} &= E\left[\sum_{i=0}^{n-1}\int_{\frac{Ti}{n}}^{\frac{T(i+1)}{n}}\left(f(B_s)-f(B_{\frac{Ti}{n}})\right)^2 ds\right]^{\frac{1}{2}} \\
&\leq \|f'\|_\infty\left(\sum_{i=0}^{n-1}\int_{\frac{Ti}{n}}^{\frac{T(i+1)}{n}}(s-\frac{Ti}{n})ds\right)^{\frac{1}{2}} \\
&\leq \frac{T\|f'\|_\infty}{\sqrt{2n}}.
\end{aligned}$$

Thus,

$$lim\int_0^{\cdot} Z_n(.,s)dB_s \underset{M^2([0,T])}{=} \lim\sum_{i=0}^{n-1} f(B_{\frac{Ti}{n}})(B_{\frac{T(i+1)}{n}\wedge .}-B_{\frac{Ti}{n}\wedge .}) \underset{M^2([0,T])}{=} \int_0^{\cdot} f(B_s)dB_s.$$

2.3 Itô Process

Definition 2.3.1 *An Itô process is a continuous and adapted process on* $[0,T]$ *of the form*

$$X_t = X_0 + \int_0^t \psi_s ds + \int_0^t \phi_s dB_s \tag{2.8}$$

where ϕ *and* ψ *belong to* $L^2_{prog}(\Omega\times[0,T])$ *and* $X_0 \in L^2(\mathcal{F}_0)$*. We will adopt the following differential notation*

$$dX_s = \psi_s ds + \phi_s dB_s.$$

Proposition 2.3.1 *The decomposition* (2.8) *is unique up to stochastic equivalence.*

Proof: This directly derives from proposition 0.5.5.□

Corollary 2.3.1 *According to proposition 0.5.5, an Itô process is a martingale if and only if the part in un ds* (ψ) *is equal to zero .*

Definition 2.3.2 *Naturally we can extend the notion of stochastic integral with respect to an Itô process. If* X *is of the form* (2.8)*, then, for* $\theta \in L^2_{prog}(\Omega \times [0,T])$ *fulfilling* $\theta\psi \in L^2_{prog}(\Omega \times [0,T])$ *and* $\theta\phi \in L^2_{prog}(\Omega \times [0,T])$ *one defines* $\int_0^t \theta_s dX_s$ *by*

$$\int_0^t \theta_s dX_s = \int_0^t \theta_s \psi_s ds + \int_0^t \theta_s \phi_s dB_s. \tag{2.9}$$

We have the following "change of variable" formula for Itô processes in the spirit of proposition 1.3.12. It may be seen as the stochastic calculus counterpart of the chain rule in ordinary calculus. The proof is left to the reader, the method being the same as in the case of the Brownian motion.

Proposition 2.3.2 *Let* X *be an Itô process of the form* (2.8)*, if* $f \in C^2(\mathbb{R},\mathbb{R})$ *with bounded derivatives, then*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)\phi_s^2 ds. \tag{2.10}$$

Remark 2.3.1 *In order to give sense to the preceding stochastic integrals we have assumed that* f *has bounded derivatives. This condition is very restrictive for applications. We will see in the next paragraph some possible extensions of the preceding formula using an other extension of the stochastic integral. Nevertheless (see exercise 1.3.3) the conditions* $f'(X)\psi \in L^2_{prog}(\Omega\times[0,T])$, $f'(X)\phi \in L^2_{prog}(\Omega\times [0,T])$ *and* $f''(X_s)\phi^2 \in L^2_{prog}(\Omega \times [0,T])$ *ensures the validity of* (2.10) *without assumptions on the derivatives of* f.

The Itô formula may easily be extended to the case of time dependent functions:

Proposition 2.3.3 *Let* X *be an Itô process of the form* (2.8)*, if* $f \in C^{(1,2)}([0,T]\times \mathbb{R},\mathbb{R})$ *with bounded derivatives, then*

$$f(t,X_t) = f(0,X_0) + \int_0^t f'_x(s,X_s)dX_s + \frac{1}{2}\int_0^t f''_{xx}(s,X_s)\phi_s^2 ds + \int_0^t f'_t(s,X_s)ds. \tag{2.11}$$

2.4 Extended Itô Calculus

In this lecture, the part concerning martingale theory is minimalist (in particular we don't deal with the notion of "stopping time") but sufficient to introduce stochastic finance and continuous time models.

The results stated in this section will be admitted and used hereafter when the integrability conditions of proposition 2.3.3 won't be fulfilled. We refer the reader to [3] for the corresponding technical proofs which are not exactly in the spirit of this introduction.

The following results are fundamental for the study of financial models (meditate them.....)

The assumption $X \in L^2_{prog}(\Omega \times [0,T])$ made in order to construct stochastic integral is sometime too restrictive. Often it can be overcome.

Define the following sets

$$\mathcal{H}^2_{loc}(\Omega \times [0,T]) = \left\{ (X_t)_{t\in[0,T]} \text{ prog meas; } \int_0^T X_s^2 ds < \infty P - a.s \right\}$$

and

$$\mathcal{H}^1_{loc}(\Omega \times [0,T]) = \left\{ (X_t)_{t\in[0,T]} \text{ prog meas; } \int_0^T |X_s| ds < \infty P - p.s \right\}.$$

Obviously we have

$$L^2_{prog}(\Omega \times [0,T]) \subset \mathcal{H}^2_{loc}(\Omega \times [0,T]) \subset \mathcal{H}^1_{loc}(\Omega \times [0,T]).$$

Proposition 2.4.1 *We can extend the stochastic integral to integrands in $\mathcal{H}^2_{loc}(\Omega\times [0,T])$. But, in this case, the stochastic process $\int_0^t X_s dB_s$* ***is not in general a martingale*** *(e.g $E[\int_0^t X_s dB_s]$ may be different from zero). Nevertheless, the properties a), b) et c) on proposition 2.2.2 remain valid.*

Remark 2.4.1 *The lack of integrability for the integrand brings about a lack of regularity for the stochastic integral. However, $\int_0^t X_s dB_s$ is know in the literature as a local martingale (it is not a martingale but not so far).*

We define generalized Itô processes

Definition 2.4.1 *A generalized Itô process is an adapted and continuous process $[0,T]$ of the form*

$$X_t = X_0 + \int_0^t \psi_s ds + \int_0^t \phi_s dB_s \tag{2.12}$$

where $\phi \in \mathcal{H}^2_{loc}(\Omega \times [0,T])$, $\psi \in \mathcal{H}^1_{loc}(\Omega \times [0,T])$ *and* X_0 *is* $\mathcal{F}^B_0$ *measurable. We will use the shorthand differential notation*

$$dX_s = \psi_s ds + \phi_s dB_s.$$

Proposition 2.4.2 *The decomposition* (2.12) *is unique:*

$$X_t = X_0 + \int_0^t \psi_s ds + \int_0^t \phi_s dB_s = X'_0 + \int_0^t \psi'_s ds + \int_0^t \phi'_s dB_s \tag{2.13}$$

implies

$$X_0 = X'_0 \ P - a.s,\ \psi = \psi' \ dx \otimes P - a.s,\ \phi = \phi' \ dx \otimes P - a.s.$$

In this framework, corollary 2.3.1 remains valid:

Proposition 2.4.3 *If* (X_t)*is a martingale of the form* 2.12 *then* $\psi = 0 \ dx \otimes P - a.s.$

The assumptions in Itô Formula become more flexible:

Proposition 2.4.4 *Let* X *be an Itô process of the form* (2.12)*. If* $f \in C^2(\mathbb{R}, \mathbb{R})$*, then,*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) \phi_s^2 ds. \tag{2.14}$$

where

$$\int_0^t f'(X_s) dX_s = \int_0^t f'(X_s) \psi_s ds + \int_0^t f'(X_s) \phi_s dB_s.$$

If f is a time dependent function we have

Proposition 2.4.5 *Let* X *be an Itô process of the form* (2.12)*. If* $f \in C^{(1,2)}([0,T] \times \mathbb{R}, \mathbb{R})$*, then,*

$$f(t, X_t) = f(0, X_0) + \int_0^t f'_x(s, X_s) dX_s + \frac{1}{2} \int_0^t f''_{xx}(s, X_s) \phi_s^2 ds + \int_0^t f'_t(s, X_s) ds. \tag{2.15}$$

Finally, when the function f is not defined on the whole real line but on an open subset Θ of $\mathbb{R}$ (e.g *log*) we have the following version:

Proposition 2.4.6 *Let X be an Itô process of the form* (2.12) *such that* $\forall t \in [0,T]$, $X_t \in \Theta\ P-a.s.$ *If* $f \in C^2(\Theta, \mathbb{R})$, *then,*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)\phi_s^2 ds. \tag{2.16}$$

where

$$\int_0^t f'(X_s)dX_s = \int_0^t f'(X_s)\psi_s ds + \int_0^t f'(X_s)\phi_s dB_s.$$

Exercise 2.4.1 *(Integration by parts formula)*
Let X and Y be two Itô processes $[0,T]$ *of the form*

$$X_t = X_0 + \int_0^t \psi_s ds + \int_0^t \phi_s dB_s$$

and

$$Y_t = Y_0 + \int_0^t \psi'_s ds + \int_0^t \phi'_s dB_s.$$

Show that

$$X_tY_t = X_0Y_0 + \int_0^t Y_s dX_s + \int_0^t X_s dY_s + \int_0^t \phi_s\phi'_s ds. \tag{2.17}$$

(Hint: We may apply Itô formula to $(X_t+Y_t)^2$, X_t^2 *and* Y_t^2*.)*

2.5 Stochastic differential equations (SDE)

2.5.1 Geometric Brownian motion

Remind that (def 1.3.1) the geometric Brownian motion with drift b and volatility σ^2 ($(b,\sigma) \in \mathbb{R}^2$) is the continuous stochastic process $(S_t)_{t\in[0,T]}$ defined by

$$S_t = x_0 e^{(b-\frac{1}{2}\sigma^2)t+\sigma B_t}. \tag{2.18}$$

Here we suppose that $x_0 > 0$ in such a way that $\forall t \in [0,T]$, $X_t > 0$. Applying the formula 2.15 with $f(t,x) = x_0 e^{(b-\frac{1}{2}\sigma^2)t+\sigma x}$ and $X_t = B_t = \int_0^t dB_s$, we obtain $\forall t \in [0,T]$

$$S_t = f(t,B_t) = f(0,B_0) + \int_0^t f'_t(s,B_s)ds + \int_0^t f'_x(s,B_s)dB_s + \frac{1}{2}\int_0^t f''_x(s,B_s)ds$$

thus

$$S_t = f(t, B_t) = x_0 + (b - \frac{1}{2}\sigma^2) \int_0^t S_s ds + \sigma \int_0^t S_s dB_s + \frac{1}{2}\sigma^2 \int_0^t S_s ds.$$

Using the differential notation (S_t) fulfills

$$\underline{\mathbf{dS_t = bS_t dt + \sigma S_t dB_t}} \tag{2.19}$$

with initial condition $S_0 = x_0$.

This so-called equation is know in finance as the **Black and Scholes** equation.

Remark 2.5.1 *According to proposition* 1.3.1*, when $b = 0$, the stochastic process S_t is a martingale. In this case, this martingale is called an exponential martingale.*

Concerning equation 2.19, the following proposition ensures the uniqueness of the solution.

Proposition 2.5.1 *For $(b, \sigma) \in \mathbb{R}^2$, there exists a unique (in the sense of definition 0.5.4) Itô process (S_t) such that*

$$dS_t = bS_t dt + \sigma S_t dB_t$$

(with $S_0 = x_0$). This process is given by 2.18.

Proof: Consider a process (X_t) fulfilling $X_0 = x_0$ and $dX_t = bX_t dt + \sigma X_t dB_t$. We put

$$Z_t = \frac{S_0}{S_t} = e^{(-b+\frac{1}{2}\sigma^2)t - \sigma B_t} = e^{(b' - \frac{1}{2}\sigma'^2)t + \sigma' B_t}$$

where $\sigma' = -\sigma$ and $b' = -b + \sigma^2$. Thus, using Itô formula,

$$Z_t = 1 + \int_0^t Z_s(b' ds + \sigma' dB_s) = 1 + \int_0^t Z_s((-b + \sigma^2)ds - \sigma dB_s).$$

From exercise 2.4.1, we deduce easily that $d(X_t Z_t) = 0$. Thus, $\forall t \in [0, T]$,

$$X_t = S_t \; P - a.s.$$

Then the process X_t is a version of S_t. These processes being continuous they are indistinguishable. □

2.5.2 General case

We refer to [6], [8] et [9] for more details on this topic.

Consider the following stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \tag{2.20}$$

with initial condition (I.C) $X_0 = Z$.

We first define the notion of solution.

Definition 2.5.1 *Let $b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and Z $\mathcal{F}_0^B$ measurable. A solution to the equation* 2.20 *is a continuous and adapted process $(X_t)_{t\in[0,T]}$ such that:*

a) $\forall t \in [0, T]$, the integrals $\int_0^t b(s, X_s)ds$ are $\int_0^t \sigma(s, X_s)dB_s$ are well defined i.e $(\sigma(t, X_t))_{t\in[0,T]} \in \mathcal{H}^2_{loc}(\Omega \times [0, T])$ and $(b(t, X_t))_{t\in[0,T]} \in \mathcal{H}^1_{loc}(\Omega \times [0, T])$.

b) $(X_t)_{t\in[0,T]}$ fulfills 2.20.

Here we establish sufficient conditions (of Lipschitz type) on b and σ to obtain an existence and uniqueness result equation 2.20. As in the case of ordinary differential equations (O.D.E) there are not necessary (see [9]).

Theorem 2.5.1 *Let b and σ two continuous functions fulfilling $\exists K > 0$ such that*

a) $|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq K|x - y|$

b) $|b(t, x)| + |\sigma(t, x)| \leq K(1 + |x|)$.

When $E[Z^2] < \infty$, equation 2.20 *has a unique solution $(X_t)_{t\in[0,T]}$ (in the sense of indistinguishability). Moreover,*

$$E[\sup_{0\leq t\leq T} |X_t|^2] < \infty.$$

Sketch of the proof: As usual when dealing with differential equations (ordinary or stochastic) the existence result use the fixed point theorem and the uniqueness one Gronwall lemma.

Step 1: Work on the "good" space: $\mathcal{S}$

We define the following complete space:

$\mathcal{S} = \{(X_t)_{t\in[0,T]};\ (X_t)_{t\in[0,T]}$ adapted and continuous such that $E[\sup_{0\le t\le T}|X_t|^2] < \infty\}$

equipped with the norm

$$\|X\|_{\mathcal{S}} = E[\sup_{0\le t\le T}|X_t|^2]^{\frac{1}{2}}$$

Step 2: Apply Picard fixed point theorem for small T

Consider the function F associating to a process $(X_t)_{t\in[0,T]}$ the process $(F(X)_t)_{t\in[0,T]}$ defined by

$$F(X)_t = Z + \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dB_s.$$

A) F is well defined

According to condition b), when $(X_t)_{t\in[0,T]} \in \mathcal{S}$, the processes $\sigma(t,X_t))_{t\in[0,T]}$ and $(b(t,X_t))_{t\in[0,T]}$ are in $L^2_{prog}(\Omega\times[0,T])$. (Note that the process $\int_0^t \sigma(s,X_s)dB_s$ is in this case a square integrable martingale).

B) $F(\mathcal{S}) \subset \mathcal{S}$

Since $(u+v)^2 \le 2(u^2+v^2)$,

$$|F(X)_t - F(0)_t|^2 \le 2(\sup_{0\le t\le T}|\int_0^t (b(s,X_s)-b(s,0))ds|^2 + \sup_{0\le t\le T}|\int_0^t(\sigma(s,X_s)-\sigma(s,0))dB_s|^2)$$

and from a)

$$E[\sup_{0\le t\le T}|\int_0^t (b(s,X_s)-b(s,0))ds|^2] \le K^2T^2 E[\sup_{0\le t\le T}|X_t|^2].$$

According to 2.4 and a)

$$E[\sup_{0\le t\le T}|\int_0^t (\sigma(s,X_s)-\sigma(s,0))dB_s|^2] \le 4K^2TE[\sup_{0\le t\le T}|X_t|^2].$$

Thus

$$\|F(X)_.\|_{\mathcal{S}} \le \sqrt{2(K^2T^2+4K^2T)}\|X_.\|_{\mathcal{S}} + \|F(0)_.\|_{\mathcal{S}}.$$

Furthermore, $(a+b+c)^2 \le 3(a^2+b^2+c^2)$ implies

$$|F(0)_t|^2 \le 3(Z^2 + \sup_{0\le t\le T}|\int_0^t b(s,0)ds|^2 + \sup_{0\le t\le T}|\int_0^t \sigma(s,0)dB_s|^2)$$

and making use of hypotheses b) and 2.4

$$\|F(0).\|_{\mathcal{S}} \leq 3(E[Z^2] + K^2T^2 + 4K^2T) < \infty.$$

The result follows because $\|F(X)\|_{\mathcal{S}} < \infty$ thus $X \in \mathcal{S}$.

C) Performing the same kind of computations as in B),

$$\|F(X) - F(Y)\|_{\mathcal{S}} \leq \sqrt{2(K^2T^2 + 4K^2T)}\|X - Y\|_{\mathcal{S}},$$

so F is Lipschitz with constant $\sqrt{2(K^2T^2 + 4K^2T)}$. If T is small enough it is even a contraction!!!

D) For small $T = T_0$, F has a unique fixed point in $\mathcal{S}$, this fixed point is clearly a solution of 2.20 on $[0, T_0]$. Thus, the solution of 2.20 on $[0, T_0]$ is unique if we restrict the space to $\mathcal{S}$.

Step 3: Uniqueness of the solution on $[0, T_0]$

This step is technical see for example [5] for a proof using the Gronwall lemma and the stopping time theorem...

Step 4: From local solutions to global solutions

We only have to successively work on $[0, T_0]$, $[T_0, 2T_0]$,.... $\square$

2.5.3 The Markov property of solutions

For $(t, x) \in ([0, T] \times \mathbb{R})$, let $(X_s^{t,x})_{s \geq t}$ be the solution of (2.20) such that $X_t = x$ thus

$$X_s^{t,x} = x + \int_t^s b(u, X_u^{t,x})du + \int_t^s \sigma(u, X_u^{t,x})dB_u.$$

The following results, giving without demonstrations, are very useful to compute conditional expectations associated to solutions of S.D.E.

Proposition 2.5.2 *In the framework of theorem* 2.5.1, *if* $s \geq t$,

$$X_s^{0,x} = X_s^{t, X_t^{0,x}} \ P.a.s. \tag{2.21}$$

Proposition 2.5.3 *In the framework of theorem* 2.5.1, *the solution of* (2.20) *is a Markov process: for all* $f : \mathbb{R} \to \mathbb{R}$ *measurable and bounded, if* $t \geq s$,

$$E[f(X_t)|\mathcal{F}_s^B] = \Phi(X_s) \ P.a.s. \tag{2.22}$$

where $\Phi(x) = E[f(X_t^{s,x})]$. *Moreover, if the coefficients* b *and* σ *don't depend on* t *(the equation is said to be homogenous)*

$$E[f(X_t)|\mathcal{F}_s^B] = \Psi(X_s) \ P.a.s. \tag{2.23}$$

where $\Psi(x) = E[f(X_{t-s}^{0,x})]$.

2.5.4 Simulation of solutions of S.D.E (a first step)

Often, S.D.E don't have explicit solutions (ndlr: closed form formula). Thus, it is interesting to use in theses cases some approximation schemes. Here we present the simplest method that is a scheme of order 1. These methods are directly derived from the one's used for O.D.E (see [1], [4] and [2]).

Stochastic Euler scheme

Here we suppose that $(X_t)_{t\in[0,T]}$ is the solution of

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

with (I.C) $X_0 = x$.

Consider the subdivision of order $N \in \mathbb{N}^*$ of the interval $[0,T]$ and put $\forall i \in \{0,...,N\}$, $t_i^N = \frac{iT}{N}$. Define the following iterative scheme: $\forall i \in \{1,...,N\}$,

$$X^N(t_i^N) = X^N(t_{i-1}^N) + b(X^N(t_{i-1}^N))\frac{T}{N} + \sigma(X^N(t_{i-1}^N))(B_{t_i^N} - B_{t_{i-1}^N}) \tag{2.24}$$

with $X^N(0) = x$. We denote by X^N the polygonal interpolation of the points of the form $(t_i^N, X^N(t_i^N))$. We have (see [2]) the following result:

Proposition 2.5.4 *In the framework of theorem* 2.5.2*,*

$$E[\sup_{t\in[0,T]} (X_t^N - X_t)^2] \leq K\frac{T}{N}$$

where K *doesn't depend on* T.

From a practical point of view, formula (2.24) is quiet simple. We only have to generate a sample (g_i) of a $\mathcal{N}(0,1)$ and substitute $\sqrt{\frac{T}{N}}g_i$ for $B_{t_i^N} - B_{t_{i-1}^N}$.

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Chapter 3

Two fundamental results

In this chapter $(B_t)_{t\in[0,T]}$ is a standard B.M defined on a probability space $(\Omega, \mathcal{A}, P)$. Moreover, this probability space is equipped with the natural brownian filtration $(\mathcal{F}_t^B)_{t\in[0,T]}$. We denote by E the expectation under P.

3.1 Girsanov's theorem

Reminder: We have already proved (exercise 0.6.1) the following result: If X is a random variable that follows, under a probability P, a $\mathcal{N}(m, \sigma^2)$ then, under the probability Q (equivalent to P) having the density

$$L = e^{-\frac{mX}{\sigma^2}} e^{+\frac{m^2}{2\sigma^2}}, \tag{3.1}$$

with respect to P, X follows a $\mathcal{N}(0, \sigma^2)$.

Here the aim is to extend this result in a dynamical way at the very least in the case of certain Gaussian processes (in particular for the Brownian motion). First we are going to deal with an elementary example.

If $m \in \mathbb{R}$, we consider the stochastic process $(\tilde{B}_t)_{t\in[0,T]}$ defined by

$$\tilde{B}_t = B_t + mt. \tag{3.2}$$

We can show easily that $(\tilde{B}_t)_{t\in[0,T]}$ is a Brownian motion under the probability P if and only if $m = 0$. Now we want to find a probability Q which makes $(\tilde{B}_t)_{t\in[0,T]}$ a standard Brownian motion under Q. Since $\tilde{B}_t$ is a $\mathcal{N}(mt, t)$, according to (3.1) we define

$$L_t = e^{-m\tilde{B}_t} e^{\frac{m^2 t}{2}} = e^{-mB_t} e^{-\frac{m^2 t}{2}}. \tag{3.3}$$

Then, one has

a) for fixed t, $\tilde{B}_t$ is, under the probability Q_t having the density L_t with respect to P, a $\mathcal{N}(0,t)$,
b) the stochastic process $(L_t)_{t\in[0,T]}$ is a martingale under P.

Finally we obtain the following proposition.

Proposition 3.1.1 *If $m \in \mathbb{R}$. Under the probability Q defined by*

$$\frac{dQ}{dP} = L_T,$$

$(\tilde{B}_t)_{t\in[0,T]}$ is a standard Brownian motion.

We need the following technical lemma.

Lemma 3.1.1 *A stochastic process $(M_t)_{t\in[0,T]}$ is martingale under Q if and only if the process $(L_tM_t)_{t\in[0,T]}$ is a martingale under P.*

Proof of the lemma: Note first that

$$E_Q[Z] = E[L_s Z]$$

if Z is $\mathcal{F}_s^B$ measurable and bounded. Moreover, if $t \geq s$ and if Y is $\mathcal{F}_s^B$ measurable and bounded,

$$E_Q[M_tY] = E[L_TM_tY] = E[E[L_TM_t|\mathcal{F}_s^B]Y] = E[E[E[L_TM_t|\mathcal{F}_t^B]|\mathcal{F}_s^B]Y]$$

thus

$$E_Q[M_tY] = E[E[L_tM_t|\mathcal{F}_s^B]Y] = E[\frac{L_s}{L_s}E[L_tM_t|\mathcal{F}_s^B]Y] = E_Q[\frac{1}{L_s}E[L_tM_t|\mathcal{F}_s^B]Y].$$

So

$$E_Q[M_t|\mathcal{F}_s^B] = \frac{1}{L_s}E[L_tM_t|\mathcal{F}_s^B].\square$$

Proof of the proposition: First we show (exercise) that $L_t\tilde{B}_t$ is a square integrable continuous martingale under P so that $L_0\tilde{B}_0 = 0$. Then, from proposition 1.3.2, we only have to prove that, $\forall\theta \in \mathbb{R}$, the process $e^{\theta\tilde{B}_t - \frac{\theta^2 t}{2}}$ is a Q martingale i.e according to the technical lemma that $L_te^{\theta\tilde{B}_t - \frac{\theta^2 t}{2}} = e^{(m+\theta)B_t - \frac{1}{2}(m+\theta)^2 t}$ is a P martingale. The last point is classically known.$\square$

The stochastic process $(L_t)_{t\in[0,T]}$ plays in this example a key role. It is called in the literature an exponential martingale. More generally, if $(\theta_t)_{t\in[0,T]} \in \mathcal{H}^2_{loc}(\Omega \times [0,T])$, one defines

$$L_t = e^{-\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds} \tag{3.4}$$

and using the Itô formula $dL_t = -L_t\theta_t dB_t$.

Exercise 3.1.1 *1) Show that $(L_t)_{t\in[0,T]}$ is a non-negative supermartingale.*
2) Show that $(L_t)_{t\in[0,T]}$ is a martingale if and only if $E[L_T] = 1$.

In this framework, we have the following generalization of the proposition 3.1.1. (cf [1])

Theorem 3.1.1 *(Girsanov) Let $(\theta_t)_{t\in[0,T]} \in \mathcal{H}^2_{loc}(\Omega \times [0,T])$ be such that the stochastic process $(L_t)_{t\in[0,T]}$ defined by*

$$L_t = e^{-\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds} \tag{3.5}$$

is a martingale under P. Then, under the probability Q with density

$$\frac{dQ}{dP} = e^{-\int_0^T \theta_s dB_s - \frac{1}{2}\int_0^T \theta_s^2 ds},$$

$(\tilde{B}_t)_{t\in[0,T]}$ where $\tilde{B}_t = B_t + \int_0^t \theta_s ds$ is a standard Brownian motion.

The following proposition (cf [1]) gives us an easy criteria to apply the preceding theorem. In fact it gives a condition for $(L_t\theta_t)_{t\in[0,T]}$ to belong to $L^2_{prog}(\Omega \times [0,T])$. This condition is sufficient because $dL_t = -L_t\theta_t dB_t$.

Proposition 3.1.2 *(Novikov) The process $(L_t)_{t\in[0,T]}$ defined by (3.5) is a martingale if*

$$E[e^{\frac{1}{2}\int_0^T \theta_s^2 ds}] < \infty.$$

3.2 Martingale representation theorem

We already know that if $(\theta_t)_{t\in[0,T]} \in L^2_{prog}(\Omega \times [0,T])$ the stochastic integral $(\int_0^t \theta_s dB_s)_{t\in[0,T]}$ is a square integrable continuous martingale relative to the Brownian filtration. In this part we are going to prove the converse: all continuous and square integrable Brownian martingales have the preceding form.

Example 3.2.1 *It is known that $((B_t)^2 - t)_{t\in[0,T]}$ and $(e^{\theta B_t - \theta^2 \frac{t}{2}})_{t\in[0,T]}$ ($\theta \in \mathbb{R}$) are $(\mathcal{F}_t^{\mathcal{B}})_{t\in[0,T]}$ martingales. Moreover, we have (example 1.3.1)*

$$B_t^2 - t = 2\int_0^t B_s dB_s$$

and according to Itô formula,

$$e^{\theta B_t - \theta^2 \frac{t}{2}} = \theta \int_0^t e^{\theta B_s - \theta^2 \frac{s}{2}} dB_s.$$

We need the following technical lemma:

Lemma 3.2.1 *The vector space $\mathcal{V}$ consists of random variables of the form*

$$e^{\int_0^T h(s)dB_s - \frac{1}{2}\int_0^T h(s)^2 ds} \tag{3.6}$$

where $h \in L^2([0,T], dx)$ is dense in $L^2(\Omega, \mathcal{F}_T^B, P)$.

Proof: Let $Y \in \mathcal{V}^\perp$, one has to show that $Y = 0$. For $(\lambda_1, ..., \lambda_p) \in \mathbb{R}^p$ and $0 \le t_1 \le ... \le t_p \le T$, (3.6) implies

$$E[Y e^{\lambda_1 B_{t_1} + ... + \lambda_p B_{t_p}}] = 0.$$

Thus, the measure defined $\forall A \in \mathcal{B}(\mathbb{R}^p)$ by

$$\mu(A) = E[Y 1_A(B_{t_1}, ..., B_{t_p})]$$

is null because its Laplace transform is null. So $\forall G \in \sigma(B_1, ..., B_{t_p})$,

$$E[Y 1_G] = 0.$$

Using the monotone class theorem, (see [1]), we have $\forall G \in \mathcal{F}_T^B$

$$E[Y 1_G] = 0.$$

Thus, $Y = 0$.□

Theorem 3.2.1 *(Itô representation theorem) let $F \in L^2(\Omega, \mathcal{F}_T^B, P)$, then there exists a unique $(\theta_t)_{t\in[0,T]} \in L^2_{prog}(\Omega \times [0,T])$ so that*

$$F = E[F] + \int_0^T \theta_s dB_s. \tag{3.7}$$

Proof: Step 1: $F \in \mathcal{V}$

If F has the form (3.6) then $E[F] = 1$ and the Itô formula ensures that

$$F = 1 + \int_0^T \left[h(u) \underbrace{e^{\int_0^u h(s)dB_s - \frac{1}{2}\int_0^u h(s)^2 ds}}_{L_u^h} \right] dB_u.$$

Moreover $(h(t)L_t^h)_{t\in[0,T]} \in L^2_{prog}(\Omega \times [0,T])$ because

$$E\left[\int_0^T (h(u)L_u^h)^2 du\right] = \int_0^T e^{\int_0^u h^2(t)dt} h^2(u)dt \le e^{\int_0^T h^2(t)dt} \int_0^T h^2(t)dt < \infty$$

thus by isometry,

$$E\left[\left(\int_0^T h(u)L_u^h dB_u\right)^2\right] = E\left[\int_0^T (h(u)L_u^h)^2 du\right].$$

So the property (3.7) is fulfilled by random variables of the form (3.6) and even (by linearity) by linear combinations of such random variables.

Step 2: General case

If $F \in L^2(\Omega, \mathcal{F}_T^B, P)$, there exists (lemma 3.2.1) a sequence $(F_n)_{n\in\mathbb{N}}$ of linear combinations of random variables of the form (3.6) that converges toward F in $L^2(\Omega, \mathcal{F}_T^B, P)$. According to the preceding step there exists a sequence $(\theta^n) \in L^2_{prog}(\Omega \times [0,T])$ such that

$$F_n = E[F_n] + \int_0^T \theta_s^n dB_s.$$

Now, we naturally (and correctly) want to pass to the limit. Using the isometry property of the Itô integral we can prove that

a) $(\theta^n)_{n\in\mathbb{N}}$ is a cauchy sequence in $L^2_{prog}(\Omega \times [0,T])$ thus (theorem 0.5.2) converges toward $\theta \in L^2_{prog}(\Omega \times [0,T])$.

b) $F = \lim_{L^2} F_n = \lim_{L^2} \left(E[F_n] + \int_0^T \theta_s^n dB_s\right) = E[F] + \int_0^T \theta_s dB_s$.

c) The representation is unique in $L^2_{prog}(\Omega \times [0,T])$.□

Theorem 3.2.2 *(Martingale representation theorem) Let $(M_t)_{t\in[0,T]}$ be a continuous and square integrable martingale (with respect to the brownian filtration), there exists a unique stochastic process $(\theta_t)_{t\in[0,T]} \in L^2_{prog}(\Omega \times [0,T])$ such that*

$$M_t = E[M_0] + \int_0^t \theta_s dB_s. \tag{3.8}$$

Proof: $M_T \in L^2(\Omega, \mathcal{F}_T^B, P)$, thus there exists $(\theta_t)_{t\in[0,T]} \in L^2_{prog}(\Omega \times [0,T])$ so that

$$M_T = \underbrace{E[M_T]}_{E[M_0]} + \int_0^T \theta_s dB_s.$$

Moreover, according to (2.6)

$$M_t = E[M_T|\mathcal{F}_t^B] = E[M_0] + E[\int_0^T \theta_s dB_s|\mathcal{F}_t^B] = E[M_0] + \int_0^t \theta_s dB_s. \square$$

Remark 3.2.1 *The preceding proof is a theoretical result of existence. Nevertheless, using the methods of Malliavin calculus (see [2]), we can often find explicitly the process* $(\theta_t)_{t\in[0,T]}$.

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Chapter 4

Applications to finance (general framework)

4.1 Introduction

4.1.1 Some historical aspects

Until the end of the sixties, the mathematical tools used concretely by financial practioners were (roughly speaking) elementary (actuarial calculus, etc...). Considering the stochastic methods exposed in the preceding chapters (rather complicated) it may be surprising to apply them to the modelling of real markets. A first question is to explain briefly this new direction and to highlight why deep links between stochastic calculus and finance have been developed...

The main explanation is the recent explosion of derivatives markets (especially options). Apparently, the first important example of such a market was the tulip bulbs market creating in XVII century in Holland. Usually, a financial market is an institution where the law of supply and demand takes place: peacefully equalize the quantity of a good demanded by consumers and the quantity supplied by producers throughout a fair price. In Holland a new type of market appeared: In order to protect themselves against unpredictable climate variations, producers of tulips developed financial contracts between two parties giving the right to fixe the price of future transactions against a certain amount of money. Unfortunately, this original experiment was very short. In fact, during a mild winter, the price of bulbs suddenly decreased, the producers exercised their rights and buyers didn't be able to pay. This example is very famous, the term tulip mania (alternatively tulipomania) is again used metaphorically to refer to any large economic bubble or crash...

Question: How to fixe the price of such a financial contract (**PRICING**) and how to use this amount of money to be able to partly overcome the risk due to randomness (**HEDGING**)?

The recent explosion of options markets, during the seventies in the USA (Chicago Board of Options Exchange in 1973) and during the eighties in Europe (Marché d'Options Négociables de Paris in 1987) may have several economical explanations. Nevertheless, it cannot be denied that the main one is link to the creation of new risk management tools directly deriving from important properties of the stochastic integral. In 1997, Scholes and Merton (see [2] and [7]) obtained their Nobel prize for this revolution (especially for works done in 1973) that provoked huge theoretical and practical changes.

4.1.2 Stochastic integral?

Here, we consider an elementary framework in order to introduce the role of stochastic integral. In particular we omit the problems of actualization.

Let us consider a time interval $[0, T]$ and an associated subdivision $0 = t_1 < \ldots t_n = T$. We suppose that (in fact it is Bachelier hypothesis see [1]) the value (i.e its price) of a financial product is given by B_t at time $t \in \{t_1, \ldots, t_n\}$. We consider a trader that adopts the following strategy: for $f \in L^2([0, T], dx)$, $\forall i \in \{1, \ldots, n-1\}$

a) he buys $f(t_i)$ assets at time t_i (the unit price is B_{t_i})

b) he sells them at time t_{i+1} (the unit price is $B_{t_{i+1}}$)

c) during the period $[t_i, t_{i+1}]$ the profit is $f(t_i)(B_{t_{i+1}} - B_{t_i})$.

On $[0, T]$ the total profit is given by

$$\sum_{i=1}^{n-1} f(t_i)(B_{t_{i+1}} - B_{t_i}).$$

If the trading is done in continuous time, this profit becomes

$$\int_0^T f(s)dB_s.$$

Using the same reasoning, it is easy to extend the preceding result in a more general setting: when the value of the stock is given by a regular Itô process $(S_t)_{t \in [0,T]}$ and when f is a non-anticipating random function (i.e a stochastic process that is progressively measurable with respect to the Brownian filtration).

Furthermore, let us consider a financial product whose price at time T (denoted by $P(T)$) is of the form

$$P(T) = k + \int_0^T f(t)dS_t. \quad (4.1)$$

If the financial market fulfills a natural hypothesis (No Arbitrage opportunity), it can be shown that the price to buy this asset at $t = 0$ is none other than k. In the same way, if the seller of such a product obtains k at time $t = 0$ and if he uses the strategy given by f, he is able to deliver the asset at time T independently of the variations of the price.

So, when P(T) is of the form 4.1, the problems of Pricing and Hedging are theoretically perfectly known. When this hypothesis is fulfilled by all the financial products (we say in this case that the market is complete), the initial question is (in theory...) entirely solved!!!

To conclude, we can remark that the representation theorem of Brownian martingales (seen in the preceding chapter) is the fundamental theoretical result that will ensure the completeness of considered markets. It is one of the secrets of the mathematics that underpins pricing models for derivative securities...

4.1.3 Is the stochastic calculus a new invisible hand?

From the preceding considerations (deep links between a powerful mathematical theory and financial applications) the reader could hastily deduce that the mystery of the Adam Smith's invisible hand (a perfect machine that naturally produces fair prices) has been revealed 2 centuries after its discovery. To conclude this paragraph let us meditate the following anecdote:

In 1997, Scholes and Merton shared the Nobel Memorial Prize in Economics. In 1994 they founded a hedge fund named Long-Term Capital Management. Initially enormously successful with annualized returns of over 40% in its first years, in 1998 it lost 4.6 billions in less than four months and became the most prominent example of the risk potential in the hedge fund industry. The fund almost folded in early 2000...([3])

4.2 Modelling financial markets in continuous time

Let us work on the period $[0,T]$ (T is called the expiration or maturity date) and on a probability space Ω that represents the space of all the possible macroeconomic configurations during this period. The space Ω is equipped with a σ-algebra $\mathcal{A}$ and with a probability P called the historical probability.

In practice the information needed for the perfect knowledge of $\mathcal{A}$ and P is not available.

4.2.1 Description of financial assets

Here we simply consider a market with two financial assets.

The non risky asset: We suppose that during the period $[0,T]$ it is possible to borrow and lend cash at a constant rate $r > 0$. Thus, if S_t^0 denotes the value at time t of one euro lent at time $t = 0$, one has, $\forall t \in [0,T]$, $S_t^0 = e^{rt}$. This asset is called the non risky asset because its value is independent from randomness.

When $(X_t)_{t\in[0,T]}$ is a stochastic process, we denote by $(\tilde{X}_t)_{t\in[0,T]}$ the associated actualized process i.e $\mathbf{\tilde{X}_t} = \frac{\mathbf{X_t}}{\mathbf{e^{rt}}}$

The risky asset: For us it will be essentially the ownership of a small piece of a company traded on a stock exchange (in opposition with over the counter markets...). The term risky comes from the fact that, contrary to the preceding one, the value of this asset depends on randomness (unpredictable macroeconomic changes). Let us denote by S_t the (random) value of this asset at time t. Thus $(S_t)_{t\in[0,T]}$ is a stochastic process adapted to its natural filtration $(\mathcal{F}_t)_{t\in[0,T]}$ ($\mathcal{F}_t = \sigma(S_u; u \leq T)$) and supposed to be continuous. The σ-algebra $\mathcal{F}_t$ represents the information available at time t.

HYP: We suppose that the dynamic of the risky asset is given by an Itô process whose value at 0 is a constant!

Remark 4.2.1 *Under the preceding hypothesis, a random variable that is $\mathcal{F}_0$ measurable is a constant.*

4.2.2 Financial strategies

HYP: no market frictions

i) There are no transaction costs or taxes.

ii) It is possible to short sell the risky asset and to borrow and lend cash.

iii) All securities are perfectly divisible (e.g. it is possible to buy 1/100th of a share).

iv) Trading is done in continuous time.

v) The risky asset does not pay a dividend.

Definition 4.2.1 *A financial strategy is a progressively measurable process* $(H_t = (\theta_t^0, \theta_t)_{t\in[0,T]})$ *with values in* $\mathbb{R}^2$ *such that the integrals*

$$\int_0^T \theta_t^0 dS_t^0 \quad and \quad \int_0^T \theta_t dS_t$$

are well defined. We associate to any financial strategy a financial portfolio containing at time $t \in [0,T]$ θ_t^0 *unities of non risky asset and* θ_t *unities of risky asset. Thus, the value at time* $t \in [0,T]$ *of such a portfolio is given by*

$$V_t^H = \theta_t^0 S_t^0 + \theta_t S_t. \tag{4.2}$$

Remark 4.2.2 *a) The measurability condition on the process* $(H_t)_{t\in[0,T]}$ *is intuitive: a decision (sell or buy) taken at* t *is naturally based on the information available at this time i.e* $\mathcal{F}_t$.

b)The fact that $(H_t)_{t\in[0,T]}$ *is indexed by* $[0,T]$*and with values in* $\mathbb{R}^2$ *is a consequence of ii), iii) and iv).*

c) (4.2) *is a consequence of i) and v).*

4.2.3 Self-financing condition

Definition 4.2.2 *A financial strategy* $(H_t)_{t\in[0,T]}$ *is said to be self-financed if the value of the associated portfolio fulfills the following S.D.E:*

$$dV_t^H = \theta_t^0 dS_t^0 + \theta_t dS_t. \tag{4.3}$$

Remark 4.2.3 *On the time interval* $[t, t+dt]$, (4.3) *implies that*

$$V_{t+dt}^H - V_t^H = \int_t^{t+dt} \theta_u^0 dS_u^0 + \int_t^{t+dt} \theta_u dS_u :$$

changes in the value of the portfolio only come from changes in the assets values. Between 0 *and* T, *we don't have the right to consume or to invest, the only possibility is to rebalance the portfolio.*

Lemma 4.2.1 *Let* $(H_t)_{t\in[0,T]}$ *and* $(H'_t)_{t\in[0,T]}$ *be two self-financed financial strategies such that* $\forall t \in [0,T]$,

$$V_t^H = V_t^{H'} \quad P.a.s$$

then H *and* H' *are equal up to indistinguishability.*

Proof: It is a direct consequence of the self-financing condition (4.3) and of proposition 2.4.2.□

Proposition 4.2.1 *A strategy is self-financed if and only if*

$$d\tilde{V}_t^H = \theta_t d\tilde{S}_t \tag{4.4}$$

Proof: Suppose that $(H_t)_{t\in[0,T]}$ is self-financed i.e $dV_t^H = \theta_t^0 dS_t^0 + \theta_t dS_t$. Since $\tilde{V}_t^H = \frac{V_t^H}{e^{rt}}$ with $d(e^{-rt}) = -re^{-rt}dt$, applying the integration by part formula (2.17)

$$d\tilde{V}_t^H = e^{-rt}dV_t^H - V_t^H re^{-rt}dt = e^{-rt}\theta_t^0 dS_t^0 + e^{-rt}\theta_t dS_t - \theta_t^0 S_t^0 re^{-rt}dt - \theta_t S_t re^{-rt}dt$$

thus,

$$d\tilde{V}_t^H = e^{-rt}\theta_t dS_t - \theta_t S_t re^{-rt}dt = \theta_t(e^{-rt}dS_t - S_t re^{-rt}dt) = \theta_t d\tilde{S}_t.$$

The reciprocal is left to the reader as an exercise.□

Remark 4.2.4 *The preceding proposition ensures that the value of a self-financed strategy only depends on the initial value and the quantity of risky asset. The quantity of non risky asset may be deduced from*

$$\tilde{V}_t^H = \theta_t^0 + \theta_t \tilde{S}_t. \tag{4.5}$$

From now on, we will use the notation $V^{x,\theta}$ *(*x *being the initial value) for* V^H.

4.2.4 Arbitrages

Definition 4.2.3 *A self financed strategy* $(H_t)_{t\in[0,T]}$ *is said to be an arbitrage opportunity (A.O) if the following conditions are fulfilled*

$$V_0^H = 0,\ V_T^H \geq 0\ P\ -a.s \text{ and } P(V_T^H > 0) > 0. \tag{4.6}$$

In other terms, starting from zero (no investment), we never lose and there are real profit opportunities.

Remark 4.2.5 *The concept of A.O depends on the choice of the historical probability P. Nevertheless, if P^* is a probability measure equivalent to P, we may replace P by P^* in the preceding definition.*

Example 4.2.1 *If you have the opportunity to buy and sell the same stock on two different markets at two different prices, there exists a trivial A.O: one can buy the less expensive one and sell the more expensive to do a riskless profit (if there are no transaction costs...). Because the differences between the prices are likely to be small (and not to last very long), this can only be done profitably with computers examining a large number of prices and automatically exercising a trade when the prices are far enough out of balance. In a "good" situation, such an opportunity should never exist.*

On financial markets, there exist practitioners called arbitrageurs paid to detect arbitrage opportunities. All things being equal, the action of these investors buying and selling to exploit the arbitrage opportunity will cause the market price of the stock to move in the direction that quickly eliminates the arbitrage (according to the law of supply and demand). In classical modelling we will always suppose that A.O never exist (at the very least for a large family of financial strategies).

4.2.5 Equivalent martingale measures (E.M.E)

Definition 4.2.4 *An E.M.M is a probability measure P^* equivalent to P under which the actualized price $(\tilde{S}_t)_{t\in[0,T]}$ of the risky asset is a martingale.*

There is a deep (and technical..) link between the existence of E.M.M and the absence of arbitrage opportunities for general continuous time models (see for example [4] for more details on this topic). In this lecture, we restrict ourselves to the following result:

Suppose that a E.M.M P^* exists.

Definition 4.2.5 *A financial strategy $(H_t)_{t\in[0,T]}$ is said to be P^* admissible if it is self-financed and if the actualized value of the associated portfolio $(\tilde{V}_t^H)$ is a martingale under P^*.*

Proposition 4.2.2 *There are no arbitrage opportunities (N.A.O) among P^* admissible financial strategies.*

Proof: According to remark 4.2.5, we may consider $P = P^*$ in the definition 4.2.3. Thus, if H is a P^* admissible financial strategy fulfilling $V_T^H \geq 0$ P^* *a.s* and $P^*(V_T^H > 0) > 0$, since $V_0^H = E[\tilde{V}_T^H|\mathcal{F}_0] = E[\tilde{V}_T^H]$, then $V_0^H > 0$ so H is not an A.O.□

Exercise 4.2.1 *If the values of two portfolios (associated to two P^* admissible financial strategies) are equal (P-a.s) at time T, show that they coincide (P-a.s) at any time $t \in [0, T]$.*

4.2.6 Contingent claims

Definition 4.2.6 *Contingent claims are assets whose prices depend on the values of other assets (for us the risky one). In our framework, it will be $\mathcal{F}_T$ measurable random variables (fulfilling some technical integrability conditions).*

Example 4.2.2 *a) A* ***european Call*** *of maturity T and strike K on the risky asset is a financial product giving the right (and not the obligation) to buy at time T a risky asset for a certain price K fixed at $t = 0$. The value at T of this product (i.e when everything is known) is equal to $Max(S_T - K, 0) := (S_T - K)_+$ (this quantity is called the payoff of the call).*

a) A ***european Put*** *of maturity T and strike K on the risky asset is a financial product giving the right (and not the obligation) to sell at time T a risky asset for a certain price K fixed at $t = 0$. The value at T of this product (i.e when everything is known) is equal to $Max(K - S_T, 0) := (K - S_T)_+$.*

These two products, giving rights to their owners, have a price (called the premium) paid at $t = 0$ that have to be determined.

4.3 Study plan and objectives

Step 1: Model the dynamic of the asset.

We have to deal with two contradictory constraints :

1) The model has to be sufficiently fine to represent the reality.

2) The model has to be sufficiently simple to be operational (cf infra).

Step 2: Study the properties of the model.

Notably, is the N.A.O condition fulfilled?

Step 3: Propose a price for a large family of contingent claims (PRICING).

Moreover, we have to be able to effectively perform this price:

a) Obtaining closed form formula with known (at least from a statistical point of view) parameters.

b) Obtaining theoretical formula that can be approximate by efficient numerical methods (discretization, Monte Carlo).

Step 4: Propose financial strategies to reduce all risks due to random fluctuations of the underlying (HEDGING).

When the price of a contingent claim is fixed, how to use the premium to ensure its delivery independently of randomness?

Step 5: (Finally....) Confront the model with reality.

1) Reality of financial markets

Several contingent claims are quoted on derivatives markets. It is especially the case for call and put options. Thus we have to

a) Compare our prices with the ones proposed by the market (calibrating the parameters to fit the data)

b) Propose prices for claims traded on over the counter markets.

2) Reality of other models

We have to compare our model with the existing ones. Some criteria may be:

a) Precision

b) Complexity (computing time...)

c) Field of applications (products that may be priced and hedged....)

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Chapter 5

The Black-Scholes model

5.1 The model

5.1.1 Dynamic of the risky asset

In the Black-Scholes model ([2]), the dynamic of the risky asset is given by the following SDE:

$$\underline{\mathbf{dS_t = bS_t dt + \sigma S_t dB_t}} \tag{5.1}$$

with initial condition $S_0 = x_0 > 0$. We have already seen (prop 2.5.1) that this SDE has a unique solution given by the geometric Brownian motion

$$S_t = x_0 e^{(b-\frac{1}{2}\sigma^2)t+\sigma B_t}. \tag{5.2}$$

Remark 5.1.1 *It is easy to deduce from the preceding formula that in the case of the Black-Scholes model the associated information filtration is none other than the Brownian filtration.*

Remark that this stochastic process is nonnegative and *a priori* only depends on two parameters b and σ respectively known as the drift and the volatility (in general the volatility is the ratio $\frac{diffusion\ coefficient}{price\ of\ the\ stock}$). This terminology is very intuitive because σ measures the sensitivity to randomness i.e to the risk and because the relation $E[S_t] = x_0 e^{bt}$ shows that, on average, the value of the stock increases as the value of a non-risky asset associated to the constant interest rate b.

Remark 5.1.2 *The question of such a choice* (5.2) *to represent the value of the stock naturally arises. To obtain this dynamic Black and Scholes have done the following hypotheses:*
1) Continuity of trajectories.
2) Stationarity of returns: the distribution of $\frac{S_{t+h}-S_t}{S_t}$ *doesn't depend on* h.

3) Independence of returns: $\frac{S_{t+h}-S_t}{S_t} \perp \mathcal{F}_t^S$.
We know (use lemma 1.1.1 with $X_t = log(S_t)$*) that these hypotheses imply a dynamic of the form* (5.2).

Moreover, it can be shown that the preceding dynamic is the one obtained passing to the limit in the so-called discrete time model of Cox, Ross and Rubinstein ([4]).

Nevertheless, we will discuss later of the limits of this hypothesis.

5.1.2 Existence of an EMM

Proposition 5.1.1 *In the Black and Scholes model there exists an EMM (def* 4.2.4*).*

Proof: Since $dS_t = bS_t dt + \sigma S_t dB_t$, we obtain from the integration by parts formula (exercise 2.4.1) that

$$d\tilde{S}_t = \tilde{S}_t(b-r)dt + \sigma \tilde{S}_t dB_t \tag{5.3}$$

and putting $W_t = B_t + \frac{b-r}{\sigma}t$,

$$d\tilde{S}_t = \sigma \tilde{S}_t dW_t. \tag{5.4}$$

According to the Girsanov theorem (prop 3.1.1), $(W_t)_{t\in[0,T]}$ is, under the probability P_0^* defined by

$$dP_0^* = e^{(\frac{b-r}{\sigma})B_T} e^{+\frac{(\frac{b-r}{\sigma})^2 T}{2}} dP, \tag{5.5}$$

a standard Brownian motion. Thus (prop 2.5.1)

$$\tilde{S}_t = \tilde{S}_0 e^{\sigma W_t - \frac{\sigma^2 t}{2}}$$

and the process $(\tilde{S}_t)_{t\in[0,T]}$ is a martingale under P_0^* (prop 1.3.1).□

Remark 5.1.3 *The probability* P_0^* *is a EMM also called a risk neutral probability. In fact, under* P_0^**,*

$$dS_t = rS_t dt + \sigma S_t dW_t$$

where W *is a standard B.M. Thus, under* P_0^**,* S_t *increases, on average, as the non-risky asset.*

Moreover we can remark that $\mathcal{F}_t^B = \mathcal{F}_t^W$*,* $\forall t \in [0,T]$.

5.2 P^* admissible financial strategies

Let P^* be a **fixed** EMM. We denote by $E^*[.]$ the expectation with respect to P^* in opposition with $E[.]$ that is the expectation with respect to the historical probability P.

5.2.1 Definition

For technical issues, we will work, from now on, with particular self-financed strategies : P^* admissible financial strategies.

Definition 5.2.1 *In the Black-Scholes model, a financial strategy H is said to be P^* admissible if it is self-financed and if the actualized value of the associated portfolio $(\tilde{V}_t^{x,\theta})_{t\in[0,T]}$ is a squared integrable and nonnegative martingale (under P^*).*

Remark 5.2.1 *Remind that in the definition of financial strategies (def 4.2.1) the integrals*

$$\int_0^T \theta_t^0 dS_t^0 = \int_0^T \theta_t^0 r e^{rt} dt \quad and \quad \int_0^T \theta_t dS_t = \int_0^T \theta_t S_t b dt + \int_0^T \sigma\theta_t S_t dB_t$$

have to be well defined.

If we suppose that

$$\theta_0 \in \mathcal{H}^1_{loc}(\Omega \times [0,T]) \tag{5.6}$$

the first one exists P-a.s (or P^*-p.p) and if we suppose that

$$\theta \in \mathcal{H}^2_{loc}(\Omega \times [0,T]), \tag{5.7}$$

(the stochastic process $(S_t)_{t\in[0,T]}$ being continuous on a compact set) then $\theta_t S_t b \in \mathcal{H}^1_{loc}(\Omega \times [0,T])$ and $\sigma\theta_t S_t \in \mathcal{H}^2_{loc}(\Omega \times [0,T])$. Thus, the second integral is well-defined.

Remark that the definitions of $\mathcal{H}^1_{loc}(\Omega \times [0,T])$ and $\mathcal{H}^2_{loc}(\Omega \times [0,T])$ is independent of the choice of P or P^*.

5.2.2 P^*-completeness of the market

Definition 5.2.2 *In this part, a contingent claim is a nonnegative random variable h in $L^2(\Omega, \mathcal{F}_T^B, P^*)$.*

Example 5.2.1 *Puts and Calls are contingent claims.*

Definition 5.2.3 *A contingent claim is said to be P^* attainable if it is equal to the final value of a portfolio associated to a strategy P^* admissible.*

Definition 5.2.4 *The financial market is P^* complete is all contingent claims are P^* attainable.*

We have the following fundamental result:

Theorem 5.2.1 *The Black and Scholes model is P^* complete. Moreover, the value at time t of **any** hedging portfolio is given by*

$$V_t = E^*[he^{-r(T-t)}|\mathcal{F}_t^B]. \tag{5.8}$$

In particular, according to proposition 2.5.3, V_t is a function of t and S_t.

Proof: Since $(\tilde{V}_t^{x,\theta})_{t\in[0,T]}$ is a martingale under P^*, the second point is obvious.
For the first one, we only give the proof in the case where $P^* = P_0^*$ (the general case being similar). We still adopt the notations introduced in the proof of the theorem 5.1.1.

Let h be a contingent claim (for P_0^*). Under the probability P_0^*, the process defined by $M_t = E^*[e^{-rT}h|\mathcal{F}_t^B]$ is a squared integrable martingale with respect to the Brownian filtration $(\mathcal{F}_t^W)$ (remark 5.1.3) thus, according to Theorem 3.2.2, there exists a stochastic process $(K_t)_{t\in[0,T]} \in L^2_{prog}(\Omega \times [0,T], P_0^*)$ such that $\forall t \in [0,T]$,

$$M_t = M_0 + \int_0^t K_s dW_s \quad P_0^* \ a.s. \tag{5.9}$$

From (5.4),

$$M_t = M_0 + \int_0^t \frac{K_s}{\sigma \tilde{S}_s} d\tilde{S}_s \quad P_0^* \ a.s, \tag{5.10}$$

and putting

$$\theta_t = \frac{K_t}{\sigma \tilde{S}_t} \quad \text{and} \quad \theta_t^0 = M_t - \theta_t \tilde{S}_t, \tag{5.11}$$

the process $H = (\theta^0, \theta)$ is a financial strategy (in the sense of definition 4.2.1) fulfilling $\forall t \in [0,T]$,

$$\tilde{V}_t^H = M_t.$$

Thus $(\tilde{V}_t^H)_{t\in[0,T]}$ is (under P_0^*) a squared integrable and nonnegative martingale. From (5.10) and proposition 4.2.1, the strategy H is self-financed. Finally, H is P_0^* admissible with $V_T^H = h$, the conclusion holds.□

5.2.3 N.A.O among P^* admissible strategies

The following result is none other than proposition 4.2.2.

Proposition 5.2.1 *There is N.A.O among P^* admissible strategies.*

Remark 5.2.2 *Note that, in general, there is no N.A.O among self-financed strategies. This particularity doesn't appear in discrete time models but is fundamental for the continuous time ones.*

5.3 Unicity of the risk neutral probability

The following result (that will be admitted) is a consequence of the P^* completeness of the Black-Scholes model and of the N.A.O among P^* admissible strategies.

Proposition 5.3.1 *In the Black-Scholes there exists a unique E.M.M (i.e P_0^*).*

Remark 5.3.1 *From now on, we will denote by P^* the unique risk neutral probability (or EMM) given by* (5.5)*. We will speak about admissible strategies, contingent claims or completeness referring implicitly to P^*.*

5.4 Pricing, Hedging

The brilliant idea of Black and Scholes was to propose the following definition for the price of a contingent claim.

Definition 5.4.1 *The price, at time $t \in [0, T]$, of a contingent claim h is the value at time t of any hedging portfolio associated to an admissible strategy (such a portfolio always exists according to Theorem* 5.2.1*). The price process will be denoted by $(P_t^h)_{t\in[0,T]}$, it is a martingale under P^*.*

Remark 5.4.1 *a) This notion is both independent of the choice of an EMM (by unicity) and of a hedging strategy (theo* 5.2.1*). Moreover, the price at time t is given by* (5.8)*. The EMM has no deep economical interpretation (contrary to the historical probability) but is a powerful tool to compute prices.*

b) In some financial model in continuous time, the unicity of the EMM is not fulfilled. For a given contingent claim several prices (in the sense of Black and Scholes) are possible. In this case we obtain a price bracket. Nevertheless, it is possible to show that the minimum of these prices is the smallest initial value of any portfolio that super-replicates the claim.

For the hedging problem, we adopt the seller's point of view. This seller proposes at $t = 0$ a financial product perfectly described by its payoff h (at T). The price of such a claim $t = 0$ is given by $E^*[he^{rT}]$. Now the following question naturally arises: How to use this amount of money in order to be able to deliver the claim at T independently of randomness? **In theory** the answer is quiet simple, we only have to build the hedging portfolio given by the Brownian martingales representation theorem (formula (5.11)). Remark that this strategy is unique by unicity of the decomposition of an Itô process (propo 2.4.2). Finally this strategy in continuous time ensures a perfect elimination of risk.

From a practical point of view, a first problem occurs. The Brownian martingales representation theorem (in the version proposed in chapter 3) is an abstract result of existence. Nevertheless, we are going to see that in important particular cases the strategy may be explicit. Moreover it can be remarked that a general answer to this important question may be found using the tools of the Malliavin calculus (see [16]).

5.5 Pricing and Hedging when $h = f(S_T)$

We are in the case of "path-independent" financial products.

Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $E^*[f^2(S_T)] < +\infty$. The value at time t of the contingent claim $h = f(S_T)$ is

$$P_t^h = E^*[e^{-r(T-t)} f(S_T)|\mathcal{F}_t^B]$$

with (under the probability P^*)

$$dS_t = rS_t dt + \sigma S_t dW_t$$

thus

$$S_t = e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}.$$

According to exercise 1.3.1 (or proposition 2.5.3), we have

$$P_t^h = e^{-r(T-t)} \int_{-\infty}^{+\infty} f(S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma y\sqrt{T-t}}) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy. \tag{5.12}$$

Thus $P_t^h = F(t, S_t)$ where

$$F(t,x) = e^{-r(T-t)} \int_{-\infty}^{+\infty} f(xe^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma y\sqrt{T-t}}) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy. \tag{5.13}$$

Remark that this formula doesn't depend on the drift coefficient b.

We have the following proposition that specify the regularity of F. In practice, this regularity is a simple consequence of changes of variables and of the Lebesgue theorem.

Lemma 5.5.1 *Under weak hypotheses (for example when h is the payoff of a Call or a Put), the function F belongs to $C^{1,2}([0,T[\times\mathbb{R},\mathbb{R})$.*

Now we are able to state the following fundamental result.

Proposition 5.5.1 *When $h = f(S_T)$, the quantity of risky asset to put in the hedging portfolio at time t is given by*

$$\theta_t = \frac{\partial F}{\partial x}(t, S_t). \tag{5.14}$$

Moreover, the function F satisfies the following PDE:

$$\frac{\partial F}{\partial t}(t,x) + rx\frac{\partial F}{\partial x}(t,x) + \frac{\sigma^2 x^2}{2}\frac{\partial^2 F}{\partial x^2}(t,x) = rF(t,x) \tag{5.15}$$

with final condition $F(T,x) = h(x)$.

Proof: Let $\tilde{F}(t,x) = e^{-rt}F(t, xe^{rt})$. Thus $\tilde{P}_t^h = \tilde{F}(t, \tilde{S}_t)$. According to the Itô formula (prop 2.4.5) (lemma 5.5.1 ensures that the hypotheses are fulfilled), one has

$$\begin{aligned} \tilde{F}(t,\tilde{S}_t) &= \tilde{F}(0,\tilde{S}_0) + \int_0^t \frac{\partial \tilde{F}}{\partial x}(u,\tilde{S}_u) \underbrace{d\tilde{S}_u}_{\sigma \tilde{S}_u dW_u} \\ &+ \int_0^t \frac{\partial \tilde{F}}{\partial t}(u,\tilde{S}_u)du + \frac{1}{2}\int_0^t \frac{\partial^2 \tilde{F}}{\partial x^2}(u,\tilde{S}_u)\sigma^2 \tilde{S}_u^2 du. \end{aligned}$$

In the same way, using the self-financing condition we obtain

$$\tilde{F}(t,\tilde{S}_t) = \tilde{F}(0,\tilde{S}_0) + \int_0^t \theta_u d\tilde{S}_u.$$

From proposition 2.4.2, we deduce

$$\theta_t = \frac{\partial \tilde{F}}{\partial x}(u,\tilde{S}_u) = \frac{\partial F}{\partial x}(t,S_t)$$

and

$$\frac{\partial \tilde{F}}{\partial t}(u,\tilde{S}_u) + \frac{1}{2}\frac{\partial^2 \tilde{F}}{\partial x^2}(u,\tilde{S}_u)\sigma^2 \tilde{S}_u^2 = 0.$$

Since

$$\frac{\partial \tilde{F}}{\partial t}(u,\tilde{S}_u) = -re^{-ru}F(u,S_u) + e^{-ru}\frac{\partial F}{\partial t}(u,S_u) + re^{-ru}S_u\frac{\partial F}{\partial x}(u,S_u)$$

and

$$\frac{\partial^2 \tilde{F}}{\partial x^2}(u,\tilde{S}_u) = e^{ru}\frac{\partial^2 F}{\partial x^2}(u,S_u)$$

then

$$-re^{-ru}F(u,S_u)+e^{-ru}\frac{\partial F}{\partial t}(u,S_u)+re^{-ru}S_u\frac{\partial F}{\partial x}(u,S_u)+\frac{1}{2}e^{-ru}\sigma^2S_u^2\frac{\partial^2 F}{\partial x^2}(u,S_u)=0.$$

The function $x_0 \in \mathbb{R}_+ \to S_u \in]0,+\infty[$ being onto, we have

$$\frac{\partial F}{\partial t}(t,x)+rx\frac{\partial F}{\partial x}(t,x)+\frac{\sigma^2x^2}{2}\frac{\partial^2 F}{\partial x^2}(t,x)=rF(t,x)$$

with $F(T,x)=h(x)$.$\square$

Remark 5.5.1 *In the Black-Scholes model, It is very interesting to notice that the prices of contingent claims may be seen as an expectation (*5.12*) or as the solution of an explicit PDE (*5.15*). From a numerical point of view this fact is remarkable because it allows to use probabilistic methods (Monte Carlo) or tools from numerical analysis (discretization schemes) to price and hedge contingent claims. This fact may be extended to more general financial models and the choice of such a method depends on the nature of the problem (regularity, dimension, etc....).*

5.6 Black and Scholes formula

This so-called formula gives the price of a Call in our framework.

Proposition 5.6.1 *The price at time t of an european call (with strike K and maturity T) is given by*

$$C_t = S_tN(d_1(t,S_t)) - Ke^{-r(T-t)}N(d_2(t,S_t)) \tag{5.16}$$

where

$$d_1(t,x)=\frac{log(\frac{x}{K})+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \text{ et } d_2(t,x)=\frac{log(\frac{x}{K})+(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{5.17}$$

and where N is the distribution function of a $\mathcal{N}(0,1)$. In this case, the composition of the hedging portfolio is given by

$$\theta_t = N(d_1(t,S_t)) > 0 \text{ and } \theta_t^0 = -Ke^{-rT}N(d_2(t,S_t)) < 0. \tag{5.18}$$

Proof: We have (5.12),

$$C_t = F(t,S_t)$$

where

$$F(t,x) = \int_{-\infty}^{+\infty} \left(x e^{-\frac{1}{2}\sigma^2(T-t)+\sigma y\sqrt{T-t}} - K e^{-r(T-t)} \right)_+ \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy. \tag{5.19}$$

But we can show that

$$x e^{-\frac{1}{2}\sigma^2(T-t)+\sigma y\sqrt{T-t}} - K e^{-r(T-t)} \geq 0 \Leftrightarrow y \geq -d_2(t,x).$$

Thus

$$\begin{aligned} F(t,x) &= \int_{-d_2}^{+\infty} \left(x e^{-\frac{1}{2}\sigma^2(T-t)+\sigma y\sqrt{T-t}} - K e^{-r(T-t)} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\ &= \int_{-\infty}^{d_2} \left(x e^{-\frac{1}{2}\sigma^2(T-t)-\sigma y\sqrt{T-t}} - K e^{-r(T-t)} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy. \end{aligned}$$

Separating the two integrals and using the change of variables $z = y + \sigma\sqrt{T-t}$ in the first one, we obtain

$$F(t,x) = xN(d_1(t,x)) - Ke^{-r(T-t)}N(d_2(t,x)).$$

Thus (5.16) and (5.18) are proved easily.□

Exercise 5.6.1 *Consider a call and a put on the risky asset with the same maturity T and the same strike K. Let us denote by C_t and P_t their prices at time $t \in [0,T]$. Using exercise 4.2.1, show that the following relation (called the call-put parity) is fulfilled:*

$$C_t + Ke^{-r(T-t)} = P_t + S_t. \tag{5.20}$$

Deduce from the preceding proposition that the price of the put at time t is given by

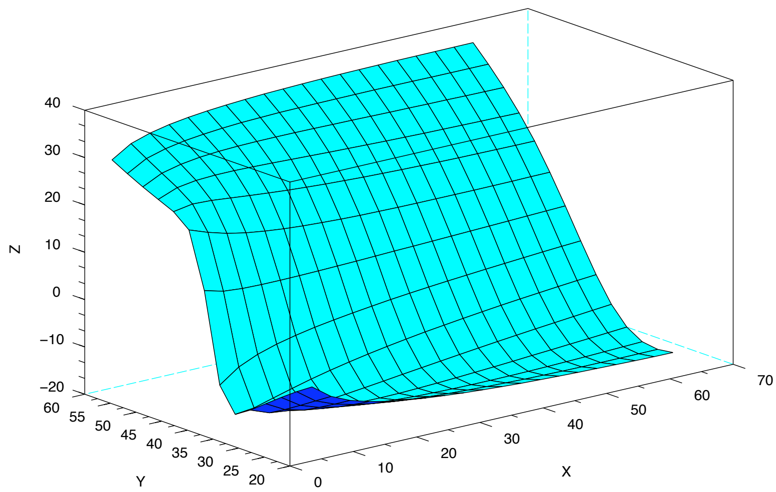
$$P_t = Ke^{-r(T-t)}N(-d_2(t,S_t)) - S_tN(-d_1(t,S_t)) \tag{5.21}$$

and that the composition of the hedging portfolio is

$$\theta_t = -N(-d_1(t,S_t)) < 0 \text{ and } \theta_t^0 = Ke^{-rT}N(-d_2(t,S_t)) > 0. \tag{5.22}$$

Remark 5.6.1 *The prices of the put and of the call are functions of $(\sigma, S_t, r, K, T-t)$. The parameter $T-t$ is called "time to maturity".*

The following graph represents the price surface $F(t,x)$ as a function of x and $100(T-t)$. We have taken $r = 9\%$, $\sigma = 30\%$, $T = 0.6$, $K = 40$.



5.7 The greeks

5.7.1 Definition

The greeks measure the sensitivity of the option prices to parameters (price of the underlying, time, volatility). In practice these quantities are fundamental. (cf infra).

When h is a contingent claim, we know that its price at time t has the form $F(t, S_t)$.

Definition 5.7.1 *We call "greeks" the following quantities:*

- Δ *measures the sensitivity of the price to the underlying*

$$\Delta_t(S_t) = \frac{\partial F}{\partial x}(t, S_t) \tag{5.23}$$

- Γ *measures the sensitivity of the delta to the underlying*

$$\Gamma_t(S_t) = \frac{\partial^2 F}{\partial x^2}(t, S_t) \tag{5.24}$$

- Θ *measures the sensitivity of the price to the time*

$$\Theta_t(S_t) = \frac{\partial F}{\partial t}(t, S_t) \tag{5.25}$$

- *ρ measures the sensitivity of the price to the interest rate*

$$\rho_t(S_t) = \frac{\partial F}{\partial r}(t, S_t) \tag{5.26}$$

- *vega (that is not a greek letter!!!) measures the sensitivity of the price to the volatility*

$$vega_t(S_t) = \frac{\partial F}{\partial \sigma}(t, S_t). \tag{5.27}$$

Using these definitions, the Black and Scholes PDE may be rewritten (in the framework of paragraph 5.5) in the following way

$$\Theta_t(x) + rx\Delta_t(x) + \frac{\sigma^2 x^2}{2}\Gamma_t(x) = rF(t,x).$$

Exercise 5.7.1 *For calls and puts, show that the values of the greeks at $t = 0$ are given by the following board:*

	Call	*Put*
Δ	$N(d_1) > 0$	$-N(-d_1) < 0$
Γ	$\frac{1}{x\sigma\sqrt{T}}N'(d_1) > 0$	$\frac{1}{x\sigma\sqrt{T}}N'(d_1) > 0$
Θ	$-\frac{x\sigma}{2\sqrt{T}}N'(d_1) - Kre^{-rT}N(d_2) < 0$	$\frac{x\sigma}{2\sqrt{T}}N'(d_1) + Kre^{-rT}(N(d_2) - 1)$??
ρ	$TKe^{-rT}N(d_2) > 0$	$TKe^{-rT}(N(d_2) - 1) < 0$
vega	$x\sqrt{T}N'(d_1) > 0$	$x\sqrt{T}N'(d_1) > 0$

Remark that in this particular case, once the parameters of the model are specified, to evaluate the greeks we only have to compute the distribution function of a Gaussian measure. This computation is not explicit (no closed form formula) but there exist good numerical approximations for this problem (see [11]). In other respects, we can also use the function "cdfnor" of scilab.

5.7.2 Payoff of the form $h = f(S_T)$

We restrict ourselves to the delta and the gamma.

When $h = f(S_T)$ we have already seen that $P_t^h = F(t, S_t)$ with

$$F(t,x) = e^{-r(T-t)}\int_{-\infty}^{+\infty} f(xe^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma y\sqrt{T-t}})\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy.$$

Proposition 5.7.1 *We have*

$$\Delta_t(x) = e^{-r(T-t)} E^* \left[\frac{W_{T-t}}{x\sigma(T-t)} f(S^x_{T-t}) \right] \tag{5.28}$$

and

$$\Gamma_t(x) = e^{-r(T-t)} E^* \left[\left(\frac{-W_{T-t}}{x^2\sigma(T-t)} + \frac{W^2_{T-t} - (T-t)}{(\sigma(T-t)x)^2} \right) f(S^x_{T-t}) \right] \tag{5.29}$$

where (S^x_t) *is (under* P^**) the geometric B.M with initial condition* $S^x_0 = x$.

Proof: We only prove (5.28), the method being the same for (5.29). We first suppose that $f \in C^1_K(\mathbb{R}, \mathbb{R})$. (For the general case we use approximations...)

Using differentiation under the integral sign, we have

$$\Delta_t(x) = e^{-r(T-t)} \int_{-\infty}^{+\infty} \frac{\partial}{\partial x} \underbrace{f(xe^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma y\sqrt{T-t}})}_{g(x,y)} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy.$$

But

$$\frac{\partial g}{\partial x}(x,y) = \frac{1}{x\sigma\sqrt{T-t}} \frac{\partial g}{\partial y}(x,y).$$

Thus by using integration by parts,

$$\Delta_t(x) = \frac{e^{-r(T-t)}}{x\sigma\sqrt{T-t}} \int_{-\infty}^{+\infty} f(xe^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma y\sqrt{T-t}}) \frac{y}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy.$$

Under P^*, $dS^x_t = rS^x_t dt + \sigma S^x_t dW_t$ so

$$S^x_t = xe^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t},$$

thus,

$$\Delta_t(x) = e^{-r(T-t)} E^* \left[\frac{W_{T-t}}{x\sigma(T-t)} f(S^x_{T-t}) \right]. \square$$

Exercise 5.7.2 *Show that*

$$vega_t(x) = x^2 T\sigma\Gamma_t(x).$$

5.7.3 Practical use of Δ and Γ

In the framework of paragraph 5.5 we have seen that $\Delta_t(S_t)$ has an important financial interpretation: It is the quantity of risky asset to put in the hedging portfolio at time t. Thus Γ represents the sensitivity of this quantity of risky asset to variations of the underlying. Thus Γ is a measure of by how much or how often a position must be rehedged in order to maintain a delta neutral position. Since in real life markets transaction costs can be large (contrary to our theoretical hypothesis) Γ is very important for the cost-effectiveness of our hedging strategy. That's why its knowledge is fundamental.

5.8 Black and Scholes in practice

(See also the Appendix)

5.8.1 Estimating volatility σ

Hedging and pricing strategies only depend on a unique and constant parameter not directly observable: the volatility (b disappears in the risk neutral universe) that is the most important and elusive quantity in the modern theory of derivatives. The problem of its estimation naturally arises.

One way to proceed is to use historical data. Let $\tilde{T} \in \mathbb{R}_+$. From observations of the dynamic of the stock in the past $[-\tilde{T}, 0]$, we can estimate σ by statistical methods. First we suppose that the dynamic in the intervall $[-\tilde{T}, 0]$ is the same as in the intervall $[0, T]$. Let $N \in \mathbb{N}^*$ be the number of observations, the random variables

$$Y_1^N = Log\left(\frac{S_0}{S_{\frac{-\tilde{T}}{N}}}\right), \ldots, Y_N^N = Log\left(\frac{S_{-\tilde{T}}}{S_{\frac{-(N-1)\tilde{T}}{N}}}\right)$$

are, under the historical probability, a N-sample of a

$$\mathcal{N}((b - \frac{\sigma^2}{2})\frac{\tilde{T}}{N}, \sigma^2\frac{\tilde{T}}{N}).$$

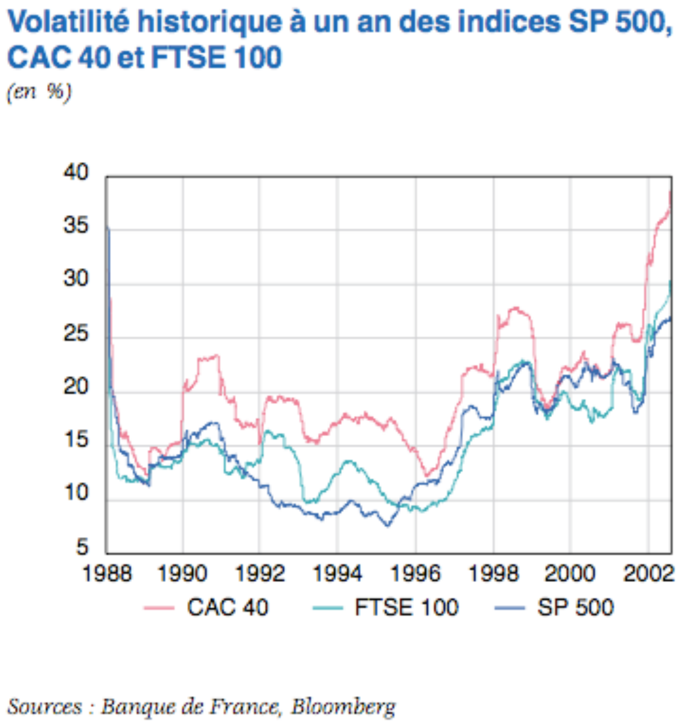
The most common estimator of σ is the empirical variance

$$\hat{\sigma} = \sqrt{\frac{N}{\tilde{T}(N-1)}\sum_{i=1}^{N}(Y_i^N - \overline{Y})^2}$$

where

$$\overline{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i^N.$$

In practice, we take $\tilde{T} = T$ to exclude the old and meaningless data. Remark that in real markets, the choice of N is limited by the frequence of the observed quotations. Moreover, more complicated procedures (GARCH models) may be used for a better understanding of this problem.



5.8.2 Pricing and Hedging

To determine the price of a financial product, the practitionner uses the formula

$$P_0^h = E^*[e^{-r(T)} f(S_T)].$$

After that, he adopts the delta hedging strategy to eliminate (in theory) the risk due to randomness. This means that the number of assets held ($\Delta_t(S_t)$ at time t for the risky one) must be continuously changed to maintain the position.

In reality, the portfolio is rehedged in discrete time because:

a) physical transactions are inevitably discrete

b) costs to buy or sell limit the number of transactions.

That's why some risks associated with the model will appear. This is one of the most important problem of continuous time modelization.

In practice on a time intervall $[t, t+h]$, the seller computes Δ in t. The question is to know if he may keep its position until $t+h$ without commiting a big hedging error. In this way, he computes Γ in t. If the absolute value of Γ is big, the seller must rebalanced the portfolio between t and $t+h$. If the Γ is small enough he may maintains its position. Remark that there is an important duality between a "good" hedging and its cost. That's why the price of a contingent claim may be seen in the following way:

Real price = Theoretic price (cost of the theoretic hedging) + transaction costs + profit marging.

Remark that the *vega* also has an important role in the precautions to take for the estimation of σ (preceding part).

5.8.3 Numerical computations of prices, of Δ and of Γ

As far as Calls and Puts are concerned, prices, Δ and Γ are given by explicit formulas (i.e closed form formulas). From a numerical point of view, we only have to compute the distribution function of a $\mathcal{N}(0,1)$ (its a classical problem, see for example the command "cdfnor" of scilab). Unfortunatly, when the payoff is more complicated such formulas are no more availlable and we have to use some approximation tools e.g Monte Carlo methods (see [3]).

Reminder: The Monte Carlo method is based on the SLLN: Let (X_n) be a sequence of i.i.d random variables such that $X \in L^1$. Putting $S_n = X_1 + ... + X_n$, we have

$$\frac{S_n}{n} \underset{a.s \text{ and } L^1}{\rightarrow} E[X_1].$$

This result provides approximate solutions to perform expectations. (Remark that the speed of convergence is given by the C.L.T).

Here we have to evaluate

$$F(t,x) = e^{-r(T-t)} E^* \left[f(S^x_{T-t}) \right], \; \Delta_t(x) = \frac{\partial F}{\partial x}(t,x) \text{ and } \Gamma_t(x) = \frac{\partial^2 F}{\partial x^2}(t,x)$$

where (S^x_t) is a geometric Brownian motion with initial condition $S^x_0 = x$.

For $F(t,x)$, we may use the classical Monte Carlo method because a N-sample of (S^x_{T-t}) is easy to simulate. For the other quantities (first and second derivatives with respect to x), one classical way is to use finite difference schemes i.e to use the following approximations (think about Taylor formula...)

$$\Delta_t(x) \approx \frac{F(t,x+h) - F(t,x-h)}{2h}$$

$$\Gamma_t(x) \approx \frac{F(t,x+h) + F(t,x-h) - 2F(t,x)}{h^2}$$

where h is small enough and where $F(t,x+h)$, $F(t,x-h)$ and $F(t,x)$ are performed by Monte Carlo method.

Problem for the greeks:

- This method embodies two different errors: discretization of the derivative function by a finite difference (choice of h???) and imperfect estimation of the option prices $F(t,x+h)$, $F(t,x-h)$ and $F(t,x)$

- In the case of a strongly discontinuous payooff function, a well known fact is its poor convergence to the exact solution.

To overcome this problem, we use proposition 5.7.1. In fact, we have proved that

$$\Delta_t(x) = e^{-r(T-t)} E^* \left[\frac{W_{T-t}}{x\sigma(T-t)} f(S^x_{T-t}) \right]$$

and

$$\Gamma_t(x) = e^{-r(T-t)} E^* \left[\left(\frac{-W_{T-t}}{x^2\sigma(T-t)} + \frac{W^2_{T-t} - (T-t)}{(\sigma(T-t)x)^2} \right) f(S^x_{T-t}) \right].$$

Thus, in the Black-Scholes model theses quantities may be performed without finite differences.

Avantages:

- Only one error (MC)
- Weight independent of the payoff
- More efficient for Γ (second derivative) than for Δ (first derivative).

Exercise 5.8.1 *Consider a* ***digital*** *charracterized by its payoff* $I_{S_T \geq K}$ *à* T.

a) Show that $F(0,x) = e^{-rT} K N(d)$, $\Delta_0(x) = \frac{e^{-rT}}{x\sigma\sqrt{T}} n(d)$ *and* $\Gamma_0(x) = \frac{e^{-rT}}{x^2\sigma^2 T} n(d) \left(d + \sigma\sqrt{T}\right)$ *where* n *is the density function of a* $\mathcal{N}(0,1)$ *and where* $d = \frac{log(\frac{x}{K})+(r-\frac{\sigma^2}{2})(T)}{\sigma\sqrt{T}}$.

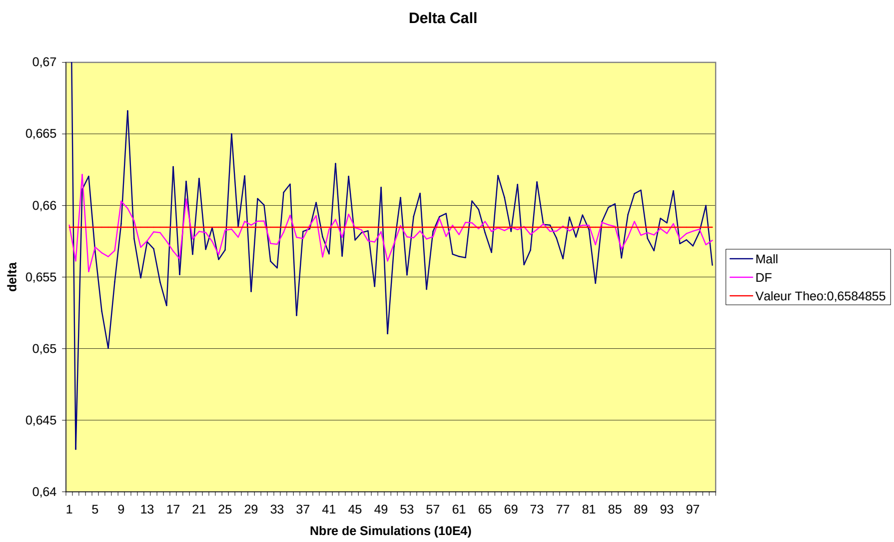
b) Using scilab, compute $\Delta_0(x)$ *and* $\Gamma_0(x)$ *using the two preceding methods. Compare the results.*

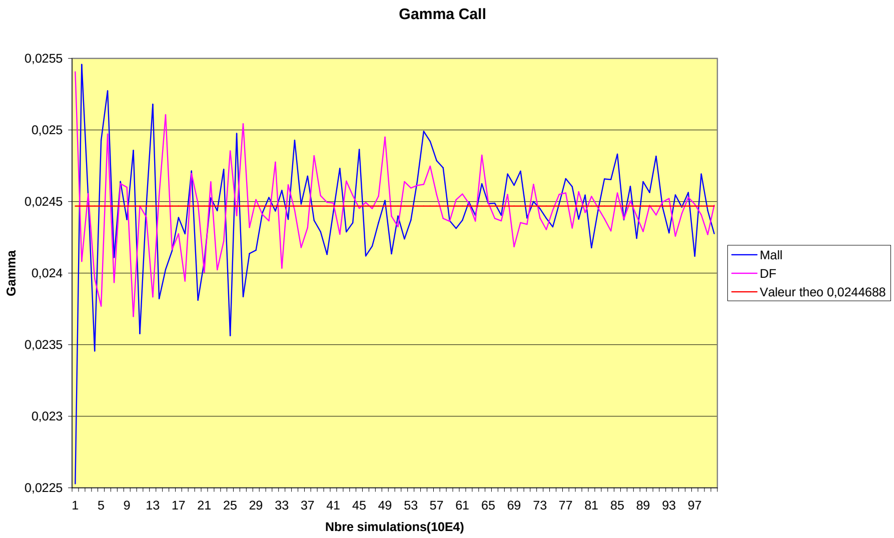
c) Answer to the preceding questions in the case of a corridor option (Payoff $I_{K_2 \geq S_T \geq K_1}$*).*

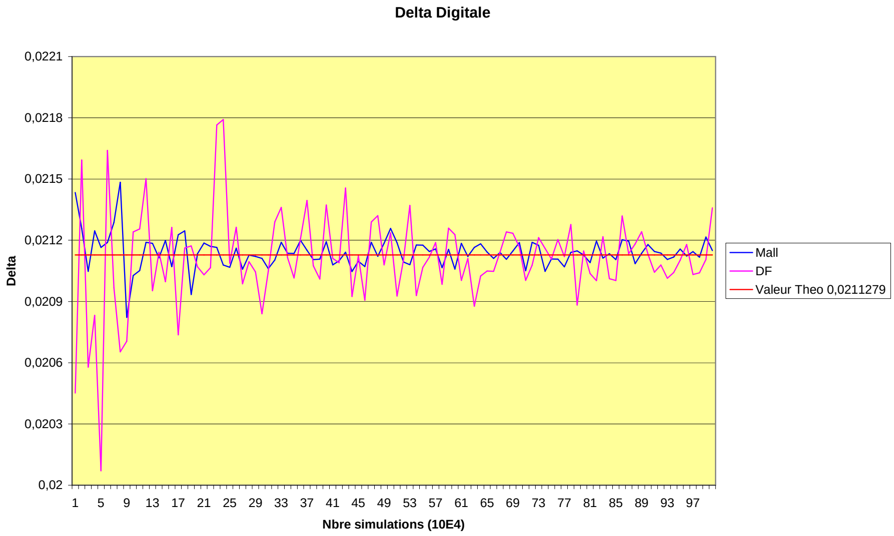
d) Define an empirical typology of option for which the weighted MC method is more efficient than the traditional finite difference one.

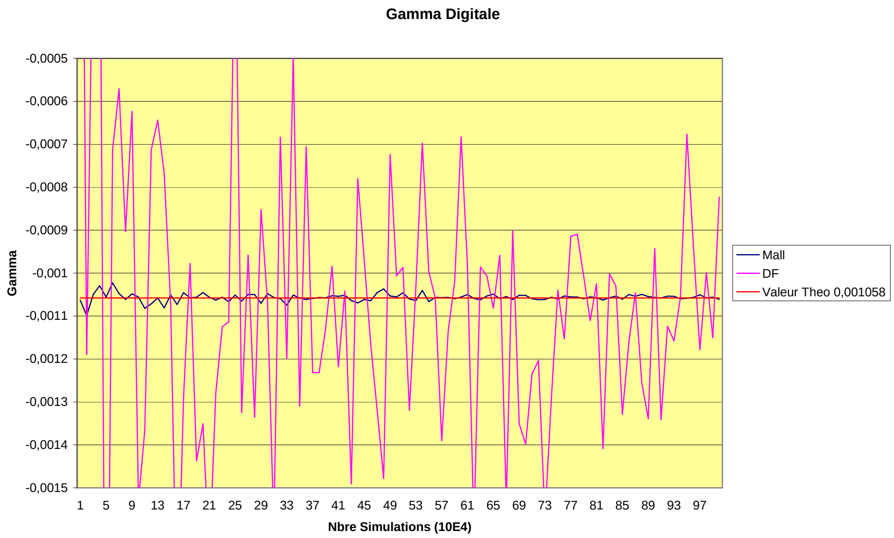
Simulations

Using the preceding notations we take$x = 100$, $K = 100$, $\sigma = 0,15$, $r = 0,05$, $T = 1$, $K_1 = 95$, $K_2 = 105$.




Gamma Call
0,0255
0,025
0,0245
0,024
0,0235
0,023
0,0225
Gamma
1 5 9 13 17 21 25 29 33 37 41 45 49 53 57 61 65 69 73 77 81 85 89 93 97
Nbre simulations(10E4)
Mall
DF
Valeur theo 0,0244688


Delta Digitale
0,0221
0,0218
0,0215
0,0212
0,0209
0,0206
0,0203
0,02
Delta
1 5 9 13 17 21 25 29 33 37 41 45 49 53 57 61 65 69 73 77 81 85 89 93 97
Nbre simulations (10E4)
Mall
DF
Valeur Theo 0,0211279


Gamma Digitale
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-0,0006
-0,0007
-0,0008
-0,0009
-0,001
-0,0011
-0,0012
-0,0013
-0,0014
-0,0015
Gamma
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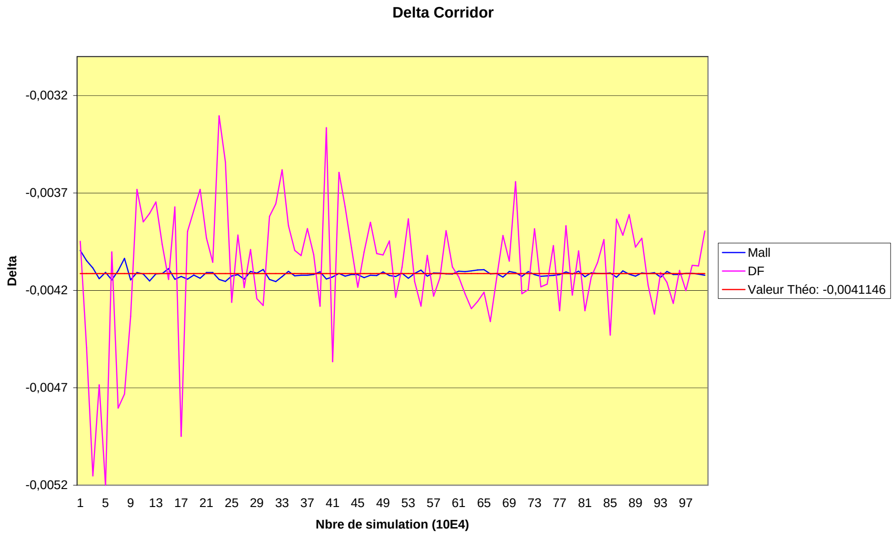
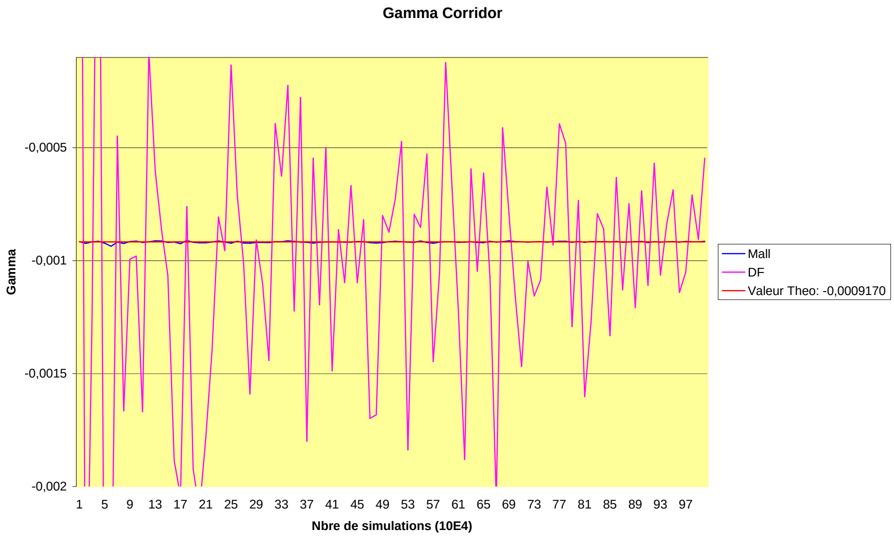
Nbre Simulations (10E4)
Mall
DF
Valeur Theo 0,001058


Delta Corridor
-0,0032
-0,0037
-0,0042
-0,0047
-0,0052
Delta
1 5 9 13 17 21 25 29 33 37 41 45 49 53 57 61 65 69 73 77 81 85 89 93 97
Nbre de simulation (10E4)
Mall
DF
Valeur Théo: -0,0041146



5.9 "Splendeurs et misères" of the Black Scholes model

In spite of some unrealistic assumptions (some of them will be discuss hereafter), the Black-Scholes model remains a fundamental tool in modern finance. The aim of this part is to introduce some important questions to continue the study beyond this elementary lecture.

5.9.1 Advantages

Options have existed for centuries, but a major constraint upon their usefulness was the enormous difficulties of their pricing. In a world where very little was known about how options should be priced, trading options was a mixture of guesswork and gambling, and very few economic agents participated in options markets. With the analytical capabilities created by Black and Scholes, the option has become a mainstream instrument ([13]), with millions of users all over the world being able to meaningfully think about option pricing (within just a few months after the Black–Scholes paper was published, Texas Instruments started selling hand calculators which had the capability of evaluating the Black–Scholes formula!!!).

From an academic point of view, Scholes (and Merton...) received the 1997 Nobel Prize in Economics for this and related work; though ineligible because of his death in 1995, the Swedish academy broke with tradition and mentioned Black as a contributor.

How to explain such a success???

Simplicity and theoretical efficiency

The Black-Scholes model only depends on a parameter not directly observable: the volatility that is a measure of the sensitivity to randomness (the stationarity of returns implying the possibility to use statistical methods to estimate this parameter). Moreover, the prices and the hedging strategies are perfectly and easily known in the case of the simplest (and fundamental) contingent claims (e.g call and put). Finally, in this framework there is (in theory cf [18]) no risk due to randomness.

Several view points

For more complicated financial products, we know that prices may only be obtained by numerical methods (no closed form formula). The Black Scholes model is interesting because two complementary approaches may be used in this case: A deterministic one linked to the approximation of solutions of P.D.E by discretization schemes and a probabilistic one based on Monte Carlo methods. We refer the reader to [7] and [19] for barrier options and to [12] and [20] for asian ones.

A self fulfilling prophecy???

Fisher Black: "*Les opérateurs savent maintenant utiliser la formule et les variantes. Ils l'utilisent tellement bien que les prix de marché sont généralement proches de ceux donnés par la formule, même lorsqu'il devrait exister un écart important...*"

The Black–Scholes formula is considered by some economists to be a "self fulfilling prophecy" in the following sense: The formula relies on several unrealistic assumptions, the most important of which is the assumption that transactions costs are zero. In reality, the trading involved in maintaining the riskless position in continuous time would involve significant transactions costs. Yet, option prices in the real world are remarkably close to those predicted by the Black–Scholes formula. One possibility is that if a sufficiently large mass of traders uses the Black–Scholes formula as a working approximation, then the formula becomes true. In this sense, it may be the case that the modern econ- omy has been steered in a certain direction because the Black–Scholes formula was discovered in 1973.

5.9.2 Limites

The log-normal hypothesis

We refer the reader to [17] for more details on this topic.

The first modern attempt at analyzing options dates back to the year 1900, when the young French mathematician Louis Bachelier wrote a dissertation at Sorbonne titled The Theory of Speculation. Bachelier was the first person to think about financial prices using the modern tools of probability theory. In particular, he thought that it was possible to model the dynamic of an asset by a "good" probability distribution that had the properties of the Brownian motion. The approach that he took and many of the results that he obtained were far ahead of their time. As a consequence, they were to lie dormant for sixty years.

Sixty five years later, Samuelson was the first to use the geometric Brownian motion (instead of the Brownian motion) to impose positivity conditions on prices.

From a technical point of view, these hypotheses (gaussian assumptions) were fruitful, in particular, they allowed the study of the equilibrium of financial markets (see Markovitz, Sharpe and Linter) that's why Black and Scholes used this framework for their option pricing theory.

Nevertheless, in the middle of the sixties, Benoit Mandelbrot published an empirical research into the distribution of cotton prices based on a very long time series which found that, contrary to the general assumption that these price movements were normally distributed, they instead followed a pareto-levy distribution. In particular, in a normally distributed market, crashes and booms are vanishingly rare, in a pareto-levy one crashes (extreme values) occur and are a significant component of the final outcome.

This important problem is still an active field of research including the study of ARCH, stochastic volatility or fractal models.

σ constant?

We refer the reader to [8] for more details on this topic.

You may consult the Web site "www.ivolatility.com" to obtain numerical data for the volatility of some american underlying.

In the Black Scholes model, the volatility is the only parameter not directly observable on the market. Since some elementary financial products are quoted on derivatives markets (e.g european calls) it is possible, using the market data, to have some knowledge on this parameter. In fact, remind that the price of a call is given by the following formula

$$C_t = S_t N(d_1(t, S_t)) - Ke^{-r(T-t)} N(d_2(t, S_t))$$

where

$$d_1(t, x) = \frac{log(\frac{x}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2(t, x) = \frac{log(\frac{x}{K}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}.$$

Thus, the price of the call is a strictly increasing function of the historical volatility because

$$\frac{\partial C}{\partial \sigma}(t, x) = x\sqrt{T-t}N'(d_1) > 0.$$

So, if we observe on the market the price at time t of a european call on the risky asset (of maturity T and strike K) (this observed price is denoted by $C_t^{Obs}(x, T, K)$), there exists a unique real number σ^{impl} such that

$$C_t^{Obs}(x, T, K) = C_t(x, T, K, \sigma^{impl}).$$

Remark 5.9.1 *The effective computation of the implicit volatility may be classically done using Newton or bisection type methods .*

In the framework of Black and Scholes, the implicit volatility should be equal to the historical one for all considered options. In practice, it is not the case. We may observed that

- The implicit volatility is greater than the historical one

- The implicit volatility both depends on the maturity and on the strike.

For the second point, the following graph shows the strike dependency of the volatility of a european call (here the underlying is quoted on the american S&P 500 market). The shape of the curve is really significant and called a smile of volatility. In fact, for $500 \leq K \leq 1400$, we can see that the function is nonincreasing (skew phenomenon) and when the strike is big it becomes nondecreasing (smile phenomenon).

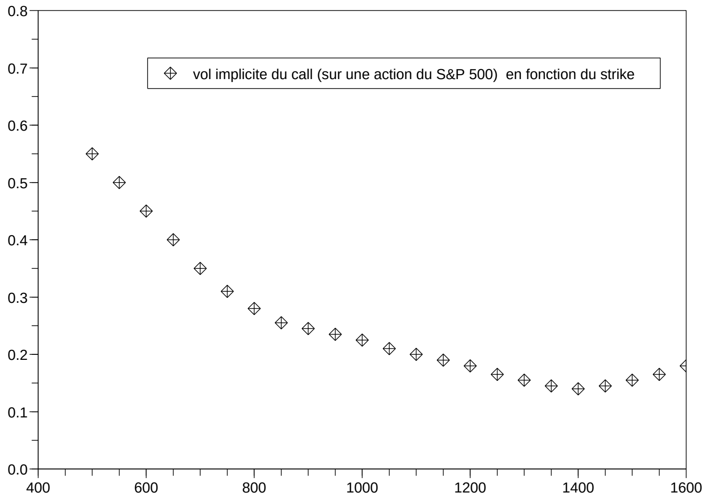


Figure 5.1: An example of volatility smile

Roughly speaking, several empirical considerations may partially highlight this fundamental phenomenon:

- The implicit volatility is bigger for the options that are out of the money ($K \neq S_0$). The market seems to give a greater probabilities to extreme values than the ones observed in a log-normal model. (links with the preceding paragraph).

- The smile may be explained by the fact that the market is less liquid for extreme strikes.

Remark 5.9.2 *Today, practitioners make their reasonings in terms of implicit volatility instead of price. Black Scholes formula has became a privileged translator.*

To conclude, remark that in spite of the preceding considerations, practioners use extensively Black-Scholes option pricing formula even to evaluate options whose underlying is known to not satisfy the Black-Scholes hypothesis of a constant volatility. In fact, when the Γ of an option is nonnegative (i.e convex payoff) it can be shown that if the trader chooses a constant volatility bigger than the real one (that may be stochastic....) the value of his hedging portfolio will always be greater then the payoff of the option at maturity (he never looses money). This fundamental property is known as the robustness of the Black and Scholes formula. In some sense it moderates the preceding criticisms.

Let us see this phenomenon in the case of a european call of maturity T and strike K.

Suppose that the price of the stock is given (under the risk neutral probability by) by

$$dS_t = rS_t dt + \Sigma_t S_t dB_t$$

where Σ_t is a regular stochastic process.

In spite of this dynamic, the practitioner computes his hedging portfolio using the Black and Scholes formula with a constant volatility σ. Thus the quantity of risky asset is given by $\Delta_{BS}(t, S_t)$ with $\Delta_{BS}(t, x) = \frac{\partial C_{BS}}{\partial x}(t, x)$ where C_{BS} is the Black-Scholes price (proposition 5.6.1). The value *at* time t of this portfolio is equal to

$$V_t = \Delta_{BS}(t, S_t)S_t + H_t^0 S_t^0$$

and the self-financing condition

$$dV_t = \Delta_{BS}(t, S_t)dS_t + H_t^0 dS_t^0$$

implies that

$$dV_t = rV_t dt + \Delta_{BS}(t, S_t)(dS_t - rS_t dt)$$

with initial condition

$$V_0 = C_{BS}(0, S_0).$$

Thus, at time T the hedging error is

$$e_T = V_T - \underbrace{C_{BS}(T, S_T)}_{(S_T - K)_+}.$$

By Itô formula,

$$dC_{BS}(t, S_t) = \left(\frac{\partial C_{BS}}{\partial t}(t, S_t) + \frac{1}{2}\frac{\partial^2 C_{BS}}{\partial x^2}(t, S_t)\Sigma_t^2 S_t^2 \right) dt + \frac{\partial C_{BS}}{\partial x}(t, S_t)dS_t.$$

But Black-Scholes PDE (5.15) ensures that

$$\frac{\partial C_{BS}}{\partial t}(t, x) + rx\frac{\partial C_{BS}}{\partial x}(t, x) + \frac{\sigma^2 x^2}{2}\frac{\partial^2 C_{BS}}{\partial x^2}(t, x) = rC_{BS}(t, x).$$

Then, we obtain the following E.D.O

$$d(V_t - C_{BS}(t, S_t)) = r(V_t - C_{BS}(t, S_t))dt - \frac{1}{2}\frac{\partial^2 C_{BS}}{\partial x^2}(t, S_t)(\Sigma_t^2 - \sigma^2)S_t^2 dt.$$

Finally

$$e_T = \frac{e^{rT}}{2}\int_0^T e^{-rt}\frac{\partial^2 C_{BS}}{\partial x^2}(t, S_t)(-\Sigma_t^2 + \sigma^2)S_t^2 dt.$$

So, When $\sigma \geq \Sigma_t$ a.s then $e_T \geq 0$ a.s, the Black-Scholes price dominates the real price.

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APPENDIX (IN FRENCH FOR THE MOMENT.......)

Simulation d'un mouvement Brownien

Simuler?

Simuler le mouvement Brownien (où plus généralement un processus stochastique) revient à calculer de manière numérique et approchée une trajectoire (ou une famille de trajectoires) de ce processus (ie $t \in [0,1] \to B_t(w)$ pour w fixé).

D'après ce qui a été dit dans le cours, plusieurs méthodes s'offrent à nous. J'en détaillerai deux.

Représentation de Wiener (1923)

Soit (g_n) une famille de $\mathcal{N}(0,1)$ indépendantes définies sur un espace $(\Omega, \mathcal{A}, P)$.

On utilise la formule suivante

$$B_t(w) = \frac{\sqrt{8}}{\pi} \sum_{n=1}^{\infty} \frac{sin(nt)}{n}\ g_n(w) \tag{30}$$

où la série converge uniformément presque sûrement sur $[0,1]$.

L'algorithme de simulation se décompose de la manière suivante:

1) La formule étant une somme infinie on doit choisir un indice k_0 suffisamment grand (par ex $k_0 = 10000$) pour valider l'approximation suivante

$$B_t(w) \simeq \frac{\sqrt{8}}{\pi} \sum_{n=1}^{k_0} \frac{sin(nt)}{n}\ g_n(w). \tag{31}$$

2) Je simule les variables aléatoires gaussiennes en utilisant la méthode de Box Muller (Je suppose ici que k_0 est pair pour la commodité de la rédaction):

a) Je fais k_0 appels à la fonction random de mon ordinateur. J'obtiens k_0 éléments de l'intervalle $[0,1]$ que je note $U_1(w)$,....,$U_{k_0}(w)$ et qui correspondent à une réalisation de k_0 variables aléatoires indépendantes distribuées suivant une $U([0,1])$.

b) $\forall j \in \{1, \frac{k_0}{2}\}$, je calcule (en faisant appel aux commandes π, cos, sin, log et $\sqrt{}$ de scilab)

$$g_{2j-1}(w) = \sqrt{-2log(U_{2j-1}(w))}cos(2\pi U_{2j}(w))$$

et

$$g_{2j}(w) = \sqrt{-2log(U_{2j-1}(w))}sin(2\pi U_{2j}(w)).$$

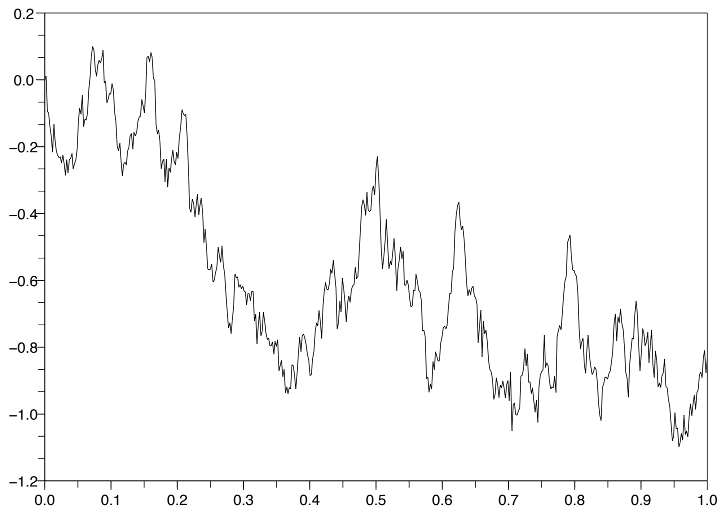
Ainsi, d'après l'exercice vu en cours les $g_1(w)$,....,$g_{k_0}(w)$ correspondent à une réalisation de k_0 variables aléatoires indépendantes distribuées suivant une $\mathcal{N}(0,1)$.

3) Je définis en faisant une boucle la fonction

$$t \to \frac{\sqrt{8}}{\pi} \sum_{n=1}^{k_0} \frac{sin(nt)}{n} \; g_n(w).$$

4) je trace cette fonction.

On obtient le résultat suivant sous scilab avec $k_0 = 1000$



Méthode de donsker
je rappelle le résultat du cours sur lequel est fondée la méthode.

On se donne une famille $(U_k)_{k\in\mathbb{N}^*}$ de variables aléatoires indépendantes, centrées et réduites. Pour tout $n \in \mathbb{N}^*$, on note $S_n = U_1 + ... + U_n$ la n-ième somme partielle. Considérons alors l'interpolation polygonale de rang n de la série des sommes partielles renormalisées: $\forall t \in [0,1]$, on pose

$$X_n(t) = \frac{1}{\sqrt{n}} \left(\sum_{k=1}^{[nt]} U_k + (nt - [nt]) U_{[nt]+1} \right). \tag{32}$$

Theorem .0.1 *La suite de processus continus* (X_n) *converge en loi dans* $\mathcal{C} = C([0,1],\mathbb{R})$ *vers la loi du M.B i.e* $\forall f \in C_b(\mathcal{C},\mathbb{R})$, $E[f(X_n)] \to E[f(B)]$.

L'algorithme de simulation est alors le suivant:

1) La formule étant une série infinie on doit choisir un indice k_0 suffisamment grand (par ex $k_0 = 10000$) pour valider l'approximation suivante:

$$B(t)(w) \approx \frac{1}{\sqrt{k_0}} \left(\sum_{k=1}^{[k_0 t]} U_k(w) + (k_0 t - [k_0 t]) U_{[k_0 t]+1}(w) \right). \tag{33}$$

2) Je fais le choix de mes variables aléatoires U_1,....,U_{k_0} centrées réduites indépendantes et je simule une réalisation de ces variables (vous pouvez utiliser si vous le voulez les gaussiennes $g_1(w)$,....,$g_{k_0}(w)$ et poser $U_j(w) = g_j(w)$, personnellement, j'ai fait un choix différent....).

3) Pour tout $j \in \{1, k_0\}$ je calcule (en faisant une boucle)

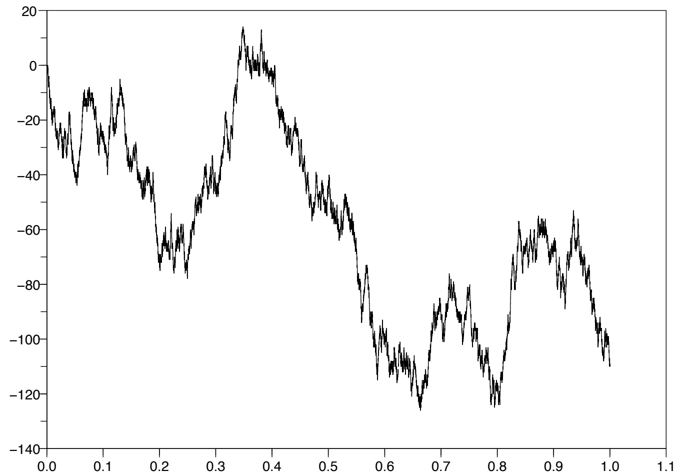
$$\frac{S_j}{\sqrt{k_0}} = \frac{U_1 + ... + U_j}{\sqrt{k_0}}.$$

4) Reste donc à tracer

$$t \to \frac{1}{\sqrt{k_0}} \left(\sum_{k=1}^{[k_0 t]} U_k(w) + (k_0 t - [k_0 t]) U_{[k_0 t]+1}(w) \right).$$

Contrairement à la forme qui peut paraître un peu barbare, ceci est en fait très simple. En effet, la fonction n'est autre que la fonction linéaire par morceaux passant par tous les points de la forme $(\frac{j}{k_0}, \frac{S_j}{\sqrt{k_0}})$. Il existe sur tous les logiciels de calcul matriciel (scilab, matlab) une fonction (appelée "plot') permettant de réaliser ceci sans travailler (et donc on en profite)!!!!

La simulation suivante a été réalisée sous scilab en choisissant les U_i de sorte que $P(U_i = 1) = P(U_i = -1) = \frac{1}{2}$ (pour la simulation des U_i je renvois aux annexes du Lamberton-Lapeyre) et en prenant $k_0 = 10000$.



Exercice 1: Essayez de le faire vous même en utilisant par exemple scilab qui est très bien et gratuit!!! On peut le télécharger à l'adresse suivante:
"http://www.scilab.org/download/index_download.php?page=release.html".

Exercice 2: Faire un travail similaire pour la méthode du point milieu.

Quelques exercices

Exercice 1: 1) Montrer que la variable aléatoire $X_t = \int_0^t cos(s)dB_s$ est bien définie.

2) Montrer que $(X_t)_{t\in[0,1]}$ est un processus gaussien dont vous calculerez la moyenne et la fonction de covariance.

3) Montrer que $X_t = cos(t)B_t + \int_0^t sin(s)B_s ds$.

Exercice 2: Soit $t \in [0,1]$ et $f \in L^2([0,1])$, montrer que

$$E[B_t \int_0^1 f(s)dB_s] = \int_0^t f(s)ds.$$

Exercice 2': On cosnidère la suite $(Z_n)_{n\in\mathbb{N}^*}$ de variables aléatoires définies par

$$Z_n = \int_0^1 (1 + \frac{s}{n})^n dB_s$$

et on pose

$$Z = \int_0^1 e^s dB_s.$$

1) Montrer que $(Z_n)_{n\in\mathbb{N}^*}$ converge vers Z dans L^2.
2) Montrer que $\forall s \in [0,1]$,

$$0 \leq s - nlog(1 + \frac{s}{n}) \leq \frac{s^2}{2n}.$$

En déduire que

$$0 \leq e^s - (1 + \frac{s}{n})^n \leq \frac{e^1}{2n}.$$

3) En déduire que pour tout $\alpha > 0$,

$$\sum_{n\geq 1} P(|Z_n - Z| > \alpha) < \infty.$$

4) Montrer que la suite $(Z_n)_{n\in\mathbb{N}^*}$ converge presque sûrement vers Z.

Exercice 3: Soit X_t la solution de l'EDS suivante:

$$dX_t = (\mu X_t + \mu')dt + (\sigma X_t + \sigma')dB_t, \quad X_0 = 0.$$

On pose $S_t = e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}$.

1) Démontrer que $d(X_t S_t^{-1}) = S_t^{-1}[(\mu' - \sigma\sigma')dt + \sigma' dB_t]$.

2) En déduire une expression pour X_t.

Exercice 4: Dans le modèle de Black et Scholes, calculer la probabilité qu'un call européen soit exercé.

Exercice 5: Soit S_t la solution de $dS_t = rS_t dt + \sigma S_t dB_t$.

1) Soit $K \in \mathbb{R}_+$, montrer que le processus

$$M_t = E\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)_+ |\mathcal{F}_t^B\right]$$

est une martingale.

2) Montrer que si l'on pose $\xi_t = S_t^{-1}\left(K - \frac{1}{T}\int_0^t S_u du\right)$, on a

$$M_t = S_t E\left[\left(\frac{1}{T}\int_t^T \frac{S_u}{S_t} du - \xi_t\right)_+ |\mathcal{F}_t^B\right].$$

3) Soit $\phi(t,x) = E\left[\left(\frac{1}{T}\int_t^T \frac{S_u}{S_t} du - x\right)_+\right]$. Montrer que $M_t = S_t \phi(t, \xi_t)$.

4) Ecrire la formule d'Itô pour M_t et en déduire une EDP vérifiée par la fonction ϕ.

Exercice 6: Soit B un MB sous une probabilité P. Soit $(M_t)_{t\in[0,1]}$ une $\mathcal{F}_t^B$ martingale (sous P) telle que $dM_t = M_t \sigma dB_t$ avec $\sigma \in \mathbb{R}$ et $M_0 = 1$.

1) Vérifier que M est à valeurs strictement positives.

2) Calculer dY_t où $Y_t = M_t^{-1}$.

3) Soit Q la probabilité définie par $\frac{dQ}{dP} = M_T$. Déterminer la loi de Y sous Q.

4) Montrer que

$$E_P[(M_T - K)_+] = K E_P[(K^{-1} - M_T)_+].$$

Exercice 7: Si X_t est solution d'une EDS de type

$$\frac{dX_t}{X_t} = r(t)dt + \sigma_t^X (dB_t + \lambda_t dt)$$

le paramètre (éventuellement aléatoire) σ^X est appelé la volatilité locale de X. Dans le cadre du modèle de Black et Scholes, montrer que la volatilité local d'un call est plus grande que la volatilité historique (ce phénomène est connu sous le nom d'effet levier).

Exercice 8: Trouver $(\theta_t)_{t\in[0,T]} \in L^2_{prog}(\Omega\times[0,T])$ tel que $F = E[F]+\int_0^T \theta_s dB_s$ pour

1) $F = B_T$,

2) $F = \int_0^T B_s ds$,

3) $F = B_T^2$,

4) $F = e^{B_T}$.

Exercice 9: On se place dans le cadre du modèle de Black et Scholes vu en cours. On note $C(S_t, t, T, K)$ (resp. $P(S_t, t, T, K)$) le prix à l'instant t du call (resp. du put) de strike K et d'échéance T.

1) Expliciter $P(S_t, t, T, K)$ en fonction de $C(S_t, t, T, K)$.

2) On fixe dorénavant $T_0 \in]0, T[$. On considère l'option DF (dépard forward) suivante: le détenteur de cette option reçoit à l'instant T_0 un call d'échéance T et de strike S_{T_0}.
a) Quel est le payoff terminal en T d'une telle option?
b) Prouver que le prix en T_0 de cette option peut s'écrire $S_{T_0}C(1, T_0, T, 1)$.
c) Montrer que le prix à $0 \le t \le T_0$ est $C(S_t, T_0, T, S_t)$.

3) On considère maintenant l'option CH (chooser) suivante: le détenteur de cette option choisit à la date T_0 si l'option en question est un call ou un put d'exercice K et d'échéance T fixés à l'avance.
a) Quel est le prix à T_0? Montrer qu'il peut se mettre sous la forme

$$C(S_{T_0}, T_0, T, K) + (Ke^{-r(T-T_0)} - S_{T_0})_+.$$

c) Trouver le prix à $0 \le t \le T_0$ de cette option. Donner un portefeuille financier statique de couverture entre 0 et T_0.

Exercice 9: Exercice 1.3.4 du poly.

Exercice 10: Soit $Y_t = h(t)dt + dB_t$ et $r_t = \sigma(t)Y_t$ où h et σ sont des fonctions de classe C^1. On souhaite calculer $E\left[e^{-\int_0^t r_s ds}\right]$.

1) Exprimer dr.
2) Soit $L_t^H = e^{\int_0^t H(s)dB_s - \frac{1}{2}\int_0^t H^2(s)ds}$. Justifier que $E[L_T^H] = 1$ pour toute fonction H continue. En déduire que

$$E\left[e^{h(T)B_T - \int_0^T h'(s)B_s ds - \frac{1}{2}\int_0^T h^2(s)ds}\right] = 1.$$

3) En utilisant L_t^h, montrer que

$$E\left[e^{-\int_0^t r_s ds}\right] = E\left[e^{h(T)B_T - \int_0^T (h'(s)+\sigma(s))B_s ds - \frac{1}{2}\int_0^T h^2(s)ds}\right].$$

4) Calculer cette quantité.

Exercice 11: On considère un actif dont la dynamique est donnée par

$$dS_t = S_t((r-\delta)dt + \sigma dB_t),\ S_0 = x$$

.

1) Montrer que dans le cadre du modèle de Black et Scholes cette dynamique modélise (sous la probabilité risque neutre Q) le prix d'un actif risqué versant des dividendes au taux continu r.

2) On souhaite évaluer un actif contingent sur S. Il s'agit donc d'évaluer

$$E_Q[h(S_T)e^{-r(T-t)}].$$

Quelle est la valeur de cet actif dans le cas où $h(x) = (x^\alpha - K)_+$.

3) On suppose $r = \delta$. On pose $d\tilde{Q} = \frac{S_T}{x}dQ$ et $Z_t = \frac{x^2}{S_t}$.

a) Justifier rapidement que l'on définit ainsi un changement de probabilité.
b) Quelle est la dynamique de Z_t sous $\tilde{Q}$?
c) Montrer que pour toute fonction borélienne bornée,

$$\frac{1}{x}E_Q[S_T f(\frac{x^2}{S_T})] = E_Q[f(S_T)].$$

4) On repasse au cas général. Montrer que le processus S^a est une martingale pour une valeur de a que l'on précisera. Montrer alors que pour toute fonction borélienne bornée,

$$\frac{1}{x^a} E_Q[S_T^a f(\frac{x^2}{S_T})] = E_Q[f(S_T)].$$

5) On se place dans le cas $h(x) = x^\beta (x - K)_+$. Montrer que $h(S_T)$ s'écrit comme différence de deux payoffs correspondants à des call européens de sous jacents $S^{\beta+1}$ et S^β avec des strikes que l'on déterminera. Conclure.

Exercice 12: Problème 1 (page 85) du Lamberton-Lapeyre.

Exercice 13: Problème 4, partie 2 (page 91) du Lamberton-Lapeyre.

Exercice 14: Problème 2, (page 86) du Lamberton-Lapeyre.

Test 10/06 (1H)

Exercice 1: Soit $(\Omega, \mathcal{A}, P)$ un espace de probabilité équipé d'une filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ et $(X_t)_{t\in\mathbb{R}_+}$ un processus réel défini sur $(\Omega, \mathcal{A}, P)$.

a) Quand dit on que le processus $(X_t)_{t\in\mathbb{R}_+}$ est adapté?

b) Quand dit on que le processus $(X_t)_{t\in\mathbb{R}_+}$ est progressivement mesurable?

c) Y a t-il un lien entre les deux notions?

d) On suppose que $(X_t)_{t\in\mathbb{R}_+}$ est adapté et continu à droite. On veut montrer que $(X_t)_{t\in\mathbb{R}_+}$ est progressivement mesurable. Pour $T \in \mathbb{R}_+$, on définit la suite $(X^n)_{n\in\mathbb{N}^*}$ de processus par $X^n_t = X_{Min(T,([\frac{nt}{T}]+1)\frac{T}{n})}$, $\forall t \in [0,T]$.

d1) Montrer que $\forall t \in [0,T]$, $X^n_t \underset{p.p}{\to} X_t$.

d2) Montrer que $\forall n \in \mathbb{N}^*$, $\forall \Gamma \in \mathcal{B}(\mathbb{R})$ on a

$$\{(t,\omega); 0 \leq t \leq T, X^n_t \in \Gamma\} \in \mathcal{B}([0,T]) \times \mathcal{F}_T.$$

d3) Conclure.

L'exercice suivant est un exercice sur les martingales pas sur le M.B

Exercice 2: Soit $(B_t)_{t\in[0,1]}$ un processus stochastique sur un espace de probabilité $(\Omega, \mathcal{A}, P)$. On note $(\mathcal{F}^{\mathcal{B}}_s = \sigma(B_u, u \leq s))_{s\in[0,1]}$ sa filtration naturelle. On suppose que les hypothèses suivantes sont vérifiées:

a) $B_0 = 0$ P-p.s.

b) $(B_t)_{t\in[0,1]}$ est continu.

b) Si $t > s$, $B_t - B_s$ est indépendant de $\mathcal{F}^{\mathcal{B}}_s$.

c) Si $t \geq s$, $B_t - B_s$ suit une $\mathcal{N}(0, t-s)$.

En utilisant les propriétés de l'espérance conditionnelle montrer que les processus $(B_t)_{t\in[0,1]}$, $((B_t)^2 - t)_{t\in[0,1]}$ et $(e^{\theta B_t - \theta^2 \frac{t}{2}})_{t\in[0,1]}$ $(\theta \in \mathbb{R})$ sont des $(\mathcal{F}^{\mathcal{B}}_t)_{t\in[0,1]}$ martingales continues de carré intégrable.

Exercice 3 Montrer que la limite de toute suite de variables aléatoires gaussiennes qui converge dans L^2 est gaussienne.

Master IRFA
Examen de Calcul Stochastique 12/06
3 Heures

Les notes de cours et les calculatrices ne sont pas autorisées. La clarté et la concision des réponses seront des éléments déterminant de la notation.

Indication de Barème: I) 6 points, II) 8 points, III) 6 points. La question III) 3 est hors barème et vaut 3 points.

Exercice I: On se place dans le cadre du modèle de Black et Scholes: la dynamique de l'actif risqué entre 0 et $T > 0$ est donnée par l'EDS suivante

$$\underline{\mathbf{dS_t = \mu S_t dt + \sigma S_t dB_t}} \tag{34}$$

de condition initiale $S_0 = x_0 > 0$ où $\mu \in \mathbb{R}$ et $\sigma \in \mathbb{R}_+^*$.

a) On pose $\forall n \in \mathbb{N}$, $\forall j \in \{0, ..., 2^n\}$, $t_j^n = \frac{Tj}{2^n}$. Montrer que

$$\lim_{n \to +\infty} \frac{1}{T} \sum_{j=0}^{2^n-1} \left(log \left(\frac{S_{t_{j+1}^n}}{S_{t_j^n}} \right) \right)^2 = \sigma^2.$$

Expliquer l'intérêt de cette relation d'un point de vue statistique.

b) On considère un actif contingent h de la forme $f(S_T)$. Rappeler la définition des grecques Δ et Γ. Quel est leur intérêt pour le praticien?

c) Expliquer en 10 lignes maximum deux points faibles du modèle de Black et Scholes.

Exercice II: Modèle de Black et Scholes avec paramètres dépendant du temps

Nous reprenons le modèle de Black et Scholes, en supposant que le prix de l'actif sans risque est donné par

$$\underline{\mathbf{dS_t^0 = r(t) S_t^0 dt;\ S_0^0 = 1}} \tag{35}$$

et celui de l'actif risqué par

$$\underline{\mathbf{dS_t = \mu(t) S_t dt + \sigma(t) S_t dB_t;\ S_0 = x_0 > 0}} \tag{36}$$

où r, μ et σ sont dans $C([0,T],\mathbb{R})$ et où B est un mouvement Brownien standard sous la probabilité historique P. On supposera de plus que $\inf_{t\in[0,T]} \sigma(t) > 0$.

1) En considérant le processus

$$Z_t = S_t e^{-\int_0^t \mu(s)ds - \int_0^t \sigma(s)dB_s + \frac{1}{2}\int_0^t \sigma^2(s)ds},$$

montrer que l'unique solution de (3) est donnée par

$$S_t = S_0 e^{\int_0^t \mu(s)ds + \int_0^t \sigma(s)dB_s - \frac{1}{2}\int_0^t \sigma^2(s)ds}.$$

2) Montrer qu'il existe une probabilité Q équivalente à P sous laquelle le prix actualisé de l'actif risqué est une martingale. Donner la densité de Q par rapport à P.

3) Soit $(H_t = (\theta_t^0, \theta_t)_{t\in[0,T]})$ une stratégie financière autofinancée dont la valeur à t est notée V_t. Montrer que lorsque $(\frac{V_t}{S_t^0})_{t\in[0,T]}$ est une martingale sous Q vérifiant $V_T = (S_T - K)_+$ alors

$$V_t = F(t, S_t)$$

où

$$F(t,x) = E_Q\left[\left(xe^{\int_t^T \sigma(s)dW_s - \frac{1}{2}\int_t^T \sigma^2(s)ds} - Ke^{-\int_t^T r(s)ds}\right)_+\right]$$

et où W est un mouvement Brownien standard sous Q.

4) Expliciter la fonction F et faire le lien avec la formule de Black et Scholes. Que peut on en déduire pour le prix d'un put d'échéance T et de strike K?

5) Construire une stratégie de couverture pour le Call d'échéance T et de strike K. Cette stratégie est elle autofinancée?

6) Démontrer que F vérifie une équation aux dérivées partielles que vous expliciterez.

Exercice III: Prix d'un zéro coupon dans le modèle de Hull et White

On considère le processus $(r(t))_{t\in[0,T]}$ vérifiant l'EDS suivante

$$dr(t) = [\alpha(t) - \beta(t)r(t)]dt + \sigma(t)dB_t \tag{37}$$

où α, β et $\sigma \in C([0,T],\mathbb{R})$. On pose $K(t) = \int_0^t \beta(s)ds$

1) Calculer $d(e^{K(t)}r(t))$. En déduire que

$$r(t) = e^{-K(t)} \left[r(0) + \int_0^t e^{K(u)} \alpha(u) du + \int_0^t e^{K(u)} \sigma(u) dB_u \right].$$

2) Montrer que $(r(t))_{t\in[0,T]}$ est un processus gaussien continu dont vous calculerez la moyenne et la fonction de covariance.

3) On pose $X_t = \int_0^t e^{K(u)} \sigma(u) dB_u$ et $Y_T = \int_0^T e^{-K(u)} X_u du$. Montrer que

$$Y_T \hookrightarrow \mathcal{N} \left(0, \int_0^T e^{2K(u)} \sigma^2(u) \left(\int_u^T e^{-2K(y)} dy \right)^2 du \right).$$

4) En déduire la loi de $\int_0^T r(s) ds$.

5) Donner le prix du zéro coupon

$$E \left[e^{-\int_0^T r(s) ds} \right].$$

Master IRFA
Calcul Stochastique 06/07
2 Heures

Les notes de cours et les calculatrices ne sont pas autorisées. La clarté et la concision des réponses seront des éléments déterminant de la notation.

Indication de Barème: I) 7 points, II) 8 points, III) 5 points, .

Exercice I: Autour du cours

1) Exposer de manière précise et concise une méthode de simulation du mouvement brownien.

2) On pose $X_t = \int_0^t cos(s)dB_s$.

2.a) Montrer que $(X_t)_{t\in[0,1]}$ est un processus gaussien dont vous calculerez la moyenne et la fonction de covariance.

2.b) Montrer que $X_t = cos(t)B_t + \int_0^t sin(s)B_s ds$.

3) Démontrer que le mouvement brownien est un processus à variations infinies sur tout intervalle.

4) En utilisant le théorème approprié vu en cours, démontrer l'existence d'une probabilité risque neutre dans le modèle de Black-Scholes

Exercice II: Moments de la solution de l'EDS de Black-Scholes

On considère l'EDS de Black-Scholes

$$\underline{\mathbf{dS_t = \mu S_t dt + \sigma S_t dB_t}} \qquad (38)$$

de condition initiale $S_0 = x_0 > 0$ où $\mu \in \mathbb{R}$ et $\sigma \in \mathbb{R}_+^*$.

a) Montrer qu'une solution de l'EDS est donnée par

$$S_t = x_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}.$$

Montrer que la solution ci-dessus est unique.

b) Calculer $E[S_t]$.

c) Pour $\alpha \geq 2$, determiner l'EDS vérifiée par S_t^α.

d) Montrer soigneusement que

$$E[S_t^\alpha] = x_0^\alpha + \int_0^t \beta E[S_s^\alpha] ds$$

où $\beta = \alpha\mu + \frac{\alpha(\alpha-1)\sigma^2}{2}$.

e) En déduire que

$$E[S_t^\alpha] = x_0^\alpha e^{\beta t}.$$

Exercice III: Changement de numéraire

On considère deux actifs financiers dont le prix est donné par les processus d'Itô suivants

$$dS_t^1 = \mu_t^1 dt + \sigma_t^1 dB_t$$
$$dS_t^2 = \mu_t^2 dt + \sigma_t^2 dB_t$$

où les intégrands sont dans $L^2_{prog}(\Omega \times [0,T])$.

Soient Φ^1 et Φ^2 deux processus stochastiques progressivement mesurables et bornés.

On définit $\forall t \in [0,T]$,

$$X_t = \Phi_t^1 S_t^1 + \Phi_t^2 S_t^2 \tag{39}$$

et on suppose que

$$dX_t = \Phi_t^1 dS_t^1 + \Phi_t^2 dS_t^2. \tag{40}$$

a) Interpréter financièrement (2) et (3).

b) Soit $(U_t)_{t\in[0,T]}$ un processus d'Itô de la forme

$$dU_t = U_t^1 dt + U_t^2 dB_t.$$

Montrer que

$$d(UX)_t = \Phi_t^1 d(US^1)_t + \Phi_t^2 d(US^2)_t.$$

c) Donner une interprétation financière simple de la question précédente.

Master IRFA
Exam of Stochastic calculus 01/08
3 Heures

Notation: 1) 3 points, 2) 11 points, 3) 8 points.

Exercise 1: Course

1) Show that a stochastic process $(B_t)_{t\in[0,1]}$ is a standard brownian motion if and only if it is a continuous and centered gaussian process with covariance function $\Gamma[s,t] = inf(s,t)$.

2) When $(B_t)_{t\in\mathbb{R}_+}$ is a brownian motion, show that $\frac{B_n}{n} \underset{n\to\infty}{\to} 0$.

Exercise 2: Black-Scholes model, binary options, options on mean

We are in the framework of the Black-Scholes model: the dynamic of the non risky asset is given by

$$\underline{\mathbf{dS^0_t = rS^0_t dt;\ S^0_0 = 1}} \tag{41}$$

and the dynamic of the risky one by

$$\underline{\mathbf{dS_t = \mu S_t dt + \sigma S_t dB_t;\ S_0 = x_0 > 0}} \tag{42}$$

where r, μ and σ are non negative real numbers and where B is a standard brownian motion under the historical probability P.

A] 1) Considering

$$Z_t = S_t e^{-\mu t - \sigma B_t + \frac{1}{2}\sigma^2 t},$$

show that the unique solution of (2) is given by

$$S_t = S_0 e^{\mu t + \sigma B_t - \frac{1}{2}\sigma^2 t}.$$

2) Remind the Black Scholes formula for the price of a european call at time $0 \le t \le T$.

B] A binary option has a payoff $I_{S_T \ge K}$ at time T.

a) Compute the price of this option at time $0 \le t \le T$.

b) Show that the associated delta and gamma are given $\forall 0 \le t \le T$ by

$$\Delta_t(x) = \frac{e^{-r(T-t)}}{x\sigma\sqrt{T-t}}n(d(t,x)) \text{ et } \Gamma_t(x) = \frac{e^{-r(T-t)}}{x^2\sigma^2(T-t)}n(d(t,x))\left(d(t,x)+\sigma\sqrt{T-t}\right)$$

where n is the density function of a $\mathcal{N}(0,1)$ and where $d(t,x) = \frac{log(\frac{x}{K})+(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$.

c) What is the interest of these quantities in finance?

d) Deduce (without computations) from a) the price of an option with a payoff equal to $I_{S_T<K}$.

C] A european call on the mean is characterized by its payoff

$$(e^{\frac{1}{T}\int_0^T ln(S_t)dt} - K)_+.$$

1) show that the price at time $t=0$ is given by

$$Z_0 N(d_1) - Ke^{-rT}N(d_2)$$

where

$$\Sigma = \frac{\sigma}{\sqrt{3}},\ Z_0 = S_0 e^{\frac{-bT}{2}-\frac{\sigma^2 T}{12}},$$

$$d_1 = \frac{ln(\frac{Z_0}{K}) + (r+\frac{\Sigma^2}{2})T}{\Sigma\sqrt{T}} \text{ and } d_2 = \frac{ln(\frac{Z_0}{K}) + (r-\frac{\Sigma^2}{2})T}{\Sigma\sqrt{T}}$$

and where N is the distribution function of a $\mathcal{N}(0,1)$.

2) What is the composition of the associated hedging portfolio?

Exercise 3: Let B be a standard brownian motion under a probability P. Let $(M_t)_{t\in[0,1]}$ be a $\mathcal{F}_t^B$ martingale (under P) such that $dM_t = M_t\sigma dB_t$ with $\sigma\in\mathbb{R}$ and $M_0 = 1$.

1) Prove that M is a strictly positive process.

2) Compute dY_t where $Y_t = M_t^{-1}$.

3) Let Q be a probability defined by $\frac{dQ}{dP} = M_T$. Find the distribution of Y under Q.

4) Show that

$$E_P[(M_T - K)_+] = KE_P[(K^{-1} - M_T)_+].$$

5) Give a financial interpretation of this result.

Master IRFA
Exam of Stochastic calculus 12/08
3 Heures

Les notes de cours et les calculatrices ne sont pas autorisées. La clarté et la concision des réponses seront des éléments déterminant de la notation.

Indication de Barème: I) 4, II) 8, III) 8.

Dans tout l'énoncé $(B_t)_{[0,T]}$ *est un mouvement brownien standard défini sur un espace* $(\Omega, \mathcal{A}, P)$*. On notera, de plus,* E *l'espérance sous* P *et* $(\mathcal{F}_t)_{[0,T]}$ *la filtration Brownienne.*

Exercice I: Autour du cours

1) Montrer que le processus $(B_t^2 - t)_{[0,T]}$ est une martingale continue par rapport à la filtration Brownienne.

2) Dans le cadre du modèle de Black-Scholes donnez (en utilisant la formule pour le call) le prix d'une option de vente à l'instant t. Quel est le portefeuille de couverture associé?

Exercice II: Ornstein-Uhlenbeck, Vasicek, CIR

On considère l'EDS

$$dX_t = \alpha X_t dt + b dB_t \tag{43}$$

où $(\alpha, b) \in \mathbb{R}^2$ et $X_0 = x \in \mathbb{R}$.

1) En utilisant le processus $Y_t = e^{-\alpha t} X_t$ montrez que **l'unique** solution de (1) est donnée par

$$X_t = e^{\alpha t} \left(x + b \int_0^t e^{-\alpha s} dB_s \right).$$

2) Montrer que (X_t) est un processus gaussien dont vous calculerez les fonctions de moyenne et de covariance.

3) Montrer que $(\int_0^t X_s ds)$ est un processus gaussien dont vous calculerez les fonctions de moyenne et de covariance.

4) Calculez $E[e^{\int_0^T X_s ds}]$.

5) En utilisant 1) montrez que l'équation

$$dZ_t = -\alpha(a - Z_t)dt + bdB_t \tag{44}$$

où $a \in \mathbb{R}$ et $Z_0 = z \in \mathbb{R}$ admet une unique solution dont vous donnerez le forme explicite.

6) On définit $\forall t \in [0, T]$, $r_t = X_t^2$. Montrez que

$$dr_t = 2\sqrt{r_t}bdB_t + (2\alpha r_t + b^2)dt.$$

Exercice III: Options asiatiques dans Black-Scholes

Nous reprenons ici le modèle de Black et Scholes, en supposant que le prix de l'actif sans risque est donné par

$$\underline{\mathbf{dS_t^0 = rS_t^0dt;\ S_0^0 = 1}} \tag{45}$$

et celui de l'actif risqué par

$$\underline{\mathbf{dS_t = \mu S_t dt + \sigma S_t dB_t;\ S_0 = x_0 > 0}} \tag{46}$$

où r, μ et σ sont des réels strictement positifs.

A] Montrez l'existence d'une probabilité Q équivalente à P telle que le processus $(\frac{S_t}{e^{rt}})_{t\in[0,T]}$ soit une martingale.

Quelle est la dynamique de (S_t) sous Q?

B] On suppose maintenant $r = 0$ et que $dS_t = \sigma S_t dW_t$ où (W_t) est un mouvement Brownien standard sous Q

1) Soit $K \in \mathbb{R}_+$, montrer que le processus

$$M_t = E_Q\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)_+ |\mathcal{F}_t^B\right]$$

est une martingale.

Que représente cette quantité d'un point de vue financier?

2) Montrer que si l'on pose $\xi_t = S_t^{-1}\left(K - \frac{1}{T}\int_0^t S_u du\right)$, on a

$$M_t = S_t E_Q\left[\left(\frac{1}{T}\int_t^T \frac{S_u}{S_t}du - \xi_t\right)_+ |\mathcal{F}_t^B\right].$$

3) Montrer pour tout $t \in [0,T]$ que $\int_t^T \frac{S_u}{S_t} du$ est indépendante de $\mathcal{F}_t^B$ et que ξ_t est $\mathcal{F}_t^B$ mesurable.

4) Soit $\phi(t,x) = E_Q\left[\left(\frac{1}{T}\int_t^T \frac{S_u}{S_t} du - x\right)_+\right]$. Montrer que $M_t = S_t \phi(t, \xi_t)$.

5) Montrer que

$$dS_t^{-1} = S_t^{-1}[(\sigma^2)dt - \sigma dB_t]$$

et que

$$d\xi_t = \xi_t[(\sigma^2)dt - \sigma dB_t] - \frac{dt}{T}.$$

6) Ecrire la formule d'Itô pour M_t et en déduire une EDP vérifiée par la fonction ϕ.